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CO-ORDINATE TRANSFORMATIONS FOR SECOND ORDER SYSTEMS. PART I: GENERAL TRANSFORMATIONS

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When the dynamics of any general second order system are cast in a state-space format, the initial choice of the state vector usually comprises one partition representing system displacements and another representing system velocities. Co-ordinate transformations can be defined which result in more general definitions of the state vector. This paper discusses the general case of co-ordinate transformations of state-space representations for second order systems. It identifies one extremely important subset of such coordinate transformations—namely the set of *structure-preserving transformations* for second order systems—and it highlights the importance of these. It shows that one particular structure-preserving transformation results in a new system characterized by real diagonal matrices and presents a forceful case that this structure-preserving transformation should be considered to be the fundamental definition for the characteristic behaviour of general second order systems—in preference to the eigenvalue–eigenvector solutions conventionally accepted.

1. INTRODUCTION

Consider the N-degree-of-freedom second order system characterized by the real $(N \times N)$ matrices $\{\mathbf{K}, \mathbf{D}, \mathbf{M}\}$ and having displacement vector $\mathbf{q}$ and force vector $\mathbf{Q}$:

$$\mathbf{K}\mathbf{q} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{M}\ddot{\mathbf{q}} = \mathbf{Q}. \tag{1}$$

Here, $\dot{\mathbf{q}}$ and $\ddot{\mathbf{q}}$ represent the first and second derivatives of the vector $\mathbf{q}$ with respect to time.

This paper is concerned with transformations to express this general system in different but equivalent forms. Because system matrices, $\{\mathbf{K}, \mathbf{D}, \mathbf{M}\}$ are not always symmetric, different transformations may be applied to the *left* and *right* of these matrices. The general case is embraced in this paper and subscripts L and R are used to distinguish *left* and *right* transformation matrices. In the special case of systems having symmetric matrices, the left and right transformations will usually be identical so that symmetry is preserved after the transformation.

When the damping is low, there is a well-founded pre-occupation with the generalized eigenvalues of the matrix pair $\{\mathbf{K}, \mathbf{M}\}$ and the associated eigenvectors. Denoting the diagonal matrix of eigenvalues as $\mathbf{\Lambda}$ and the corresponding matrices of *left* and *right* eigenvectors as $\{\mathbf{U}_L, \mathbf{U}_R\}$, respectively, a general definition of these quantities can be written as

$$\mathbf{\Lambda}^{-1}\mathbf{U}_L^{\mathrm{T}}\mathbf{K} = \mathbf{U}_L^{\mathrm{T}}\mathbf{M}, \qquad \mathbf{K}\mathbf{U}_R\mathbf{\Lambda}^{-1} = \mathbf{M}\mathbf{U}_R. \tag{2}$$

If all of the eigenvalues are distinct, it is simple to show that with appropriate scaling of the eigenvectors, equation (2) leads directly to

$$\mathbf{U}_L^{\mathrm{T}}\mathbf{K}\mathbf{U}_R = \mathbf{\Lambda}, \qquad \mathbf{U}_L^{\mathrm{T}}\mathbf{M}\mathbf{U}_R = \mathbf{I}. \tag{3}$$

In some cases where the eigenvalues are not distinct, it is not possible to find full-rank $(N \times N)$ matrices $\{\mathbf{U}_L, \mathbf{U}_R\}$ satisfying equation (2) and it follows directly that equation (3) cannot be satisfied in these cases. Such systems are referred to as *defective* systems.

There is one other exception to equation (3). This occurs when $\mathbf{M}$ is singular. It is straightforward to provide for this by recognizing that equation (3) essentially describes a diagonalizing similarity transformation in which the transformed stiffness matrix is $\mathbf{\Lambda}$ and the transformed mass matrix is $\mathbf{I}$. If a general scaling is allowed for the eigenvectors, equation (3) generalizes to

$$\mathbf{U}_L^{\mathrm{T}}\mathbf{K}\mathbf{U}_R = \mathbf{K}_D, \qquad \mathbf{U}_L^{\mathrm{T}}\mathbf{M}\mathbf{U}_R = \mathbf{M}_D, \tag{4}$$

where $\{\mathbf{K}_D, \mathbf{M}_D\}$ are diagonal matrices of the transformed system. Equation (2) generalizes to

$$\tilde{\mathbf{K}}_D\mathbf{U}_L^{\mathrm{T}}\mathbf{K} = \tilde{\mathbf{M}}_D\mathbf{U}_L^{\mathrm{T}}\mathbf{M} \qquad \mathbf{K}\mathbf{U}_R\tilde{\mathbf{K}}_D = \mathbf{M}\mathbf{U}_R\tilde{\mathbf{M}}_D. \tag{5}$$

where $\{\tilde{\mathbf{K}}_D, \tilde{\mathbf{M}}_D\}$ are diagonal matrices *dual* to $\{\mathbf{K}_D, \mathbf{M}_D\}$ in the sense that they satisfy

$$\tilde{\mathbf{K}}_D\mathbf{K}_D = \tilde{\mathbf{M}}_D\mathbf{M}_D. \tag{6}$$

One suitable and obvious choice of the *dual* system is obtained simply through

$$\tilde{\mathbf{K}}_D = \mathbf{M}_D, \qquad \tilde{\mathbf{M}}_D = \mathbf{K}_D. \tag{7}$$

The diagonalizing transformation (of equation (4)) is possible only where the system is not defective. When $\{\mathbf{K}, \mathbf{M}\}$ are both symmetric and when either one of them is positive semi-definite, then $\{\mathbf{U}_L, \mathbf{U}_R, \mathbf{K}_D, \mathbf{M}_D\}$ are real $(N \times N)$ matrices and one choice of scaling leads to $\mathbf{U}_L = \mathbf{U}_R$ [1]. In other cases, $\{\mathbf{U}_L, \mathbf{U}_R, \mathbf{K}_D, \mathbf{M}_D\}$ may sometimes contain complex numbers.

A system is described as *classically damped* if the same transformation $\{\mathbf{U}_L, \mathbf{U}_R\}$ that diagonalizes the mass and stiffness matrices also diagonalizes the damping matrix. Caughey and O'Kelly [2] discuss this in the context of self-adjoint systems ($\mathbf{K} = \mathbf{K}^{\mathrm{T}}$, $\mathbf{D} = \mathbf{D}^{\mathrm{T}}$, $\mathbf{M} = \mathbf{M}^{\mathrm{T}}$). The extension to the general case may be considered to be a definition for the purposes of this paper. For any *classically damped* system $\{\mathbf{K}, \mathbf{D}, \mathbf{M}\}$, there is some transformation, $\{\mathbf{U}_L, \mathbf{U}_R\}$, such that

$$\begin{aligned} \tilde{\mathbf{K}}_D\mathbf{U}_L^{\mathrm{T}}\mathbf{K} = \tilde{\mathbf{D}}_D\mathbf{U}_L^{\mathrm{T}}\mathbf{D} = \tilde{\mathbf{M}}_D\mathbf{U}_L^{\mathrm{T}}\mathbf{M}, \\ \mathbf{K}\mathbf{U}_R\tilde{\mathbf{K}}_D = \mathbf{D}\mathbf{U}_R\tilde{\mathbf{D}}_D = \mathbf{M}\mathbf{U}_R\tilde{\mathbf{M}}_D, \end{aligned} \tag{8}$$

where $\{\tilde{\mathbf{K}}_D, \tilde{\mathbf{D}}_D, \tilde{\mathbf{M}}_D\}$ are diagonal matrices *dual* to $\{\mathbf{K}_D, \mathbf{D}_D, \mathbf{M}_D\}$ in the sense that they satisfy

$$\tilde{\mathbf{K}}_D\mathbf{K}_D = \tilde{\mathbf{D}}_D\mathbf{D}_D = \tilde{\mathbf{M}}_D\mathbf{M}_D \tag{9}$$

with $\{\mathbf{K}_D, \mathbf{D}_D, \mathbf{M}_D\}$ being the diagonal matrices of the transformed system. In this case, an obvious choice for the duals arises as

$$\tilde{\mathbf{K}}_D = \mathbf{D}_D\mathbf{M}_D, \qquad \tilde{\mathbf{D}}_D = \mathbf{K}_D\mathbf{M}_D, \qquad \tilde{\mathbf{M}}_D = \mathbf{K}_D\mathbf{D}_D. \tag{10}$$

If this classically damped system is not defective,

$$\mathbf{U}_L^{\mathrm{T}}\mathbf{K}\mathbf{U}_R = \mathbf{K}_D, \qquad \mathbf{U}_L^{\mathrm{T}}\mathbf{D}\mathbf{U}_R = \mathbf{D}_D, \qquad \mathbf{U}_L^{\mathrm{T}}\mathbf{M}\mathbf{U}_R = \mathbf{M}_D. \tag{11}$$

Together $\{\mathbf{U}_L, \mathbf{U}_R\}$ describe a transformation from the original set of displacement co-ordinates, $\mathbf{q}$, and its corresponding vector of forces, $\mathbf{Q}$, to a new set of displacement co-ordinates $\mathbf{r}$ and the corresponding vector of forces $\mathbf{R}$ through

$$\mathbf{q} = \mathbf{U}_R\mathbf{r}, \qquad \mathbf{R} = \mathbf{U}_L^{\mathrm{T}}\mathbf{Q}. \tag{12}$$

Then, the original equation of motion for a classically damped system is transformed to

$$\mathbf{K}_D\mathbf{r} + \mathbf{D}_D\dot{\mathbf{r}} + \mathbf{M}_D\ddot{\mathbf{r}} = \mathbf{R}. \tag{13}$$

Because the equations in equation (13) are completely decoupled, the combination of equations (12) and (13) provides for the very efficient calculation of response in the time or frequency domains through the use of superposition. It also provides for a clear understanding of the mechanisms through which the system responds (especially when $\{\mathbf{U}_L, \mathbf{U}_R\}$ are real). The left modal matrix, $\mathbf{U}_L$, acts to transform physical forces into corresponding modal forces and the right modal matrix, $\mathbf{U}_R$, acts to transform modal displacements into physical displacements.

For systems that are not classically damped, the situation is not nearly so clear using present-day methods. In general, there is no pair of $(N \times N)$ matrices $\{\mathbf{U}_L, \mathbf{U}_R\}$ (real or complex) that can simultaneously diagonalize the three system matrices according to equation (11).

The original system can be represented as a system of first order differential equations in state-space form. In this case, the two *system* matrices in the state-space equation each have dimension $(2N \times 2N)$ and the inherent second order nature of the original system is effectively ignored. The $2N$ characteristic roots and their associated $2N$ modal vectors (left and right) can be computed but, in general, these are complex and their full significance is difficult to grasp [3].

Many researchers have battled with the implications of complex modes in various contexts including:

- interpretation of complex modes [1,3–7];
- the search for iterative or approximate solutions for the damped natural frequencies and for system response using nearby classically damped models [8–12];
- model correlation, model updating and system identification [13–19];
- model reduction of generally damped systems [20–23].

The first priority of this paper is to show that real-valued transformations do exist for most real second order systems such that system response can be assembled as the direct sum of contributions from N decoupled single-degree-of-freedom second order systems. These transformations exist for all real second order systems having no repeated pairs of characteristic roots and they are referred to here as *diagonalizing structure-preserving* transformations.

It is then natural to consider whether there are more general *structure-preserving* transformations for second order systems. The second priority of the paper is to show that there are and that the *diagonalizing structure-preserving transformation* can be constructed as the product of a number of the more general *structure-preserving* transformations.

2. REAL DIAGONALIZING TRANSFORMATIONS FOR GENERAL SECOND ORDER SYSTEMS

Define

$$\mathbf{p} := \dot{\mathbf{q}}, \qquad \mathbf{P} := \dot{\mathbf{Q}}. \tag{14}$$

Then, it is possible to write equation (1) equivalently in any of the following three forms:

$$\begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \mathbf{p} \end{bmatrix} - \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \mathbf{Q}, \tag{15}$$

$$\begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \mathbf{p} \end{bmatrix} + \begin{bmatrix} \mathbf{D} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \end{bmatrix} \mathbf{Q}, \tag{16}$$

$$\begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \mathbf{p} \end{bmatrix} + \begin{bmatrix} \mathbf{D} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \ddot{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} \mathbf{P} \\ \mathbf{Q} \end{bmatrix}. \tag{17}$$

Solutions to equation (1) must satisfy all three of equations (15)–(17).

There are only three different $(2N \times 2N)$ matrices in equations (15)–(17). Ordinarily, the characteristic behaviour of the system described in equation (1) is computed by solving a generalized eigenvalue problem defined either by the two $(2N \times 2N)$ matrices in equation (15) or by the two $(2N \times 2N)$ matrices in equation (16). The latter is usually preferred. The result is a set of characteristic roots (eigenvalues from the generalized eigenvalue problem) and associated characteristic vectors. It is usual that many, if not all, of the characteristic roots are complex in which case they and their associated characteristic vectors occur in complex conjugate pairs. Solving the generalized eigenvalue problem defined by the two $(2N \times 2N)$ matrices of equation (17) yields the squares of these characteristic roots.

An alternative equivalent expression of the characteristic behaviour of the system is achievable in which only real transformation matrices appear [24]. A modified version of this expression is given here. Define

$$\underline{\mathbf{K}} := \begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K} & \mathbf{D} \end{bmatrix}, \qquad \underline{\mathbf{D}} := \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix}, \qquad \underline{\mathbf{M}} := \begin{bmatrix} \mathbf{D} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \tag{18}$$

for the sake of compactness in later expressions. Additionally, define the following quantities based on diagonal matrices $\{\mathbf{K}_D, \mathbf{D}_D, \mathbf{M}_D\}$:

$$\underline{\mathbf{K}_D} := \begin{bmatrix} \mathbf{0} & \mathbf{K}_D \\ \mathbf{K}_D & \mathbf{D}_D \end{bmatrix}, \qquad \underline{\mathbf{D}_D} := \begin{bmatrix} \mathbf{K}_D & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_D \end{bmatrix}, \qquad \underline{\mathbf{M}_D} := \begin{bmatrix} \mathbf{D}_D & \mathbf{M}_D \\ \mathbf{M}_D & \mathbf{0} \end{bmatrix} \tag{19}$$

and note the perfect similarity in structure between equations (18) and (19).

Equation (8) applies only to *classically damped* systems. Its generalization to the set of all second order systems can be written as

$$\begin{aligned} \underline{\tilde{\mathbf{K}}_D}\underline{\mathbf{U}_L^{\mathrm{T}}}\underline{\mathbf{K}} &= \underline{\tilde{\mathbf{D}}_D}\underline{\mathbf{U}_L^{\mathrm{T}}}\underline{\mathbf{D}} = \underline{\tilde{\mathbf{M}}_D}\underline{\mathbf{U}_L^{\mathrm{T}}}\underline{\mathbf{M}}, \\ \underline{\mathbf{K}}\underline{\mathbf{U}_R}\underline{\tilde{\mathbf{K}}_D} &= \underline{\mathbf{D}}\underline{\mathbf{U}_R}\underline{\tilde{\mathbf{D}}_D} = \underline{\mathbf{M}}\underline{\mathbf{U}_R}\underline{\tilde{\mathbf{M}}_D}, \end{aligned} \tag{20}$$

where $\{\underline{\tilde{\mathbf{K}}_D}, \underline{\tilde{\mathbf{D}}_D}, \underline{\tilde{\mathbf{M}}_D}\}$ are real diagonal matrices *dual* to $\{\underline{\mathbf{K}_D}, \underline{\mathbf{D}_D}, \underline{\mathbf{M}_D}\}$ in the sense that they satisfy

$$\underline{\tilde{\mathbf{K}}_D}\underline{\mathbf{K}_D} = \underline{\tilde{\mathbf{D}}_D}\underline{\mathbf{D}_D} = \underline{\tilde{\mathbf{M}}_D}\underline{\mathbf{M}_D} \tag{21}$$

and $\{\underline{\mathbf{U}_L}, \underline{\mathbf{U}_R}\}$are real $(2N \times 2N)$ matrices obviously having properties that are very like matrices of *left* and *right* eigenvectors but these are not matrices of eigenvectors. An obvious choice for the duals is

$$\underline{\tilde{\mathbf{K}}_D} := \begin{bmatrix} -\mathbf{M}_D\mathbf{D}_D\mathbf{M}_D & \mathbf{M}_D\mathbf{K}_D\mathbf{M}_D \\ \mathbf{M}_D\mathbf{K}_D\mathbf{M}_D & \mathbf{0} \end{bmatrix}, \qquad \underline{\tilde{\mathbf{D}}_D} := \begin{bmatrix} \mathbf{M}_D\mathbf{K}_D\mathbf{M}_D & \mathbf{0} \\ \mathbf{0} & -\mathbf{K}_D\mathbf{M}_D\mathbf{K}_D \end{bmatrix},$$

$$\underline{\tilde{\mathbf{M}}_D} := \begin{bmatrix} \mathbf{0} & \mathbf{K}_D\mathbf{M}_D\mathbf{K}_D \\ \mathbf{K}_D\mathbf{M}_D\mathbf{K}_D & -\mathbf{K}_D\mathbf{D}_D\mathbf{K}_D \end{bmatrix} \tag{22}$$

These are obtained by finding the inverses of $\{\underline{\mathbf{K}_D}, \underline{\mathbf{D}_D}, \underline{\mathbf{M}_D}\}$, respectively, and multiplying each individual $(N \times N)$ block by $(\mathbf{K}_D^2\mathbf{M}_D^2)$.

Theorem *If there is no pair of integers (i,j) for which it is possible to find real scalar, α, satisfying*

$$\mathbf{K}_D(i,\, i) = \alpha\mathbf{K}_D(j,\, j), \quad \mathbf{D}_D(i, i) = \alpha\mathbf{D}_D(j,\, j), \quad \mathbf{M}_D(i,\, i) = \alpha\mathbf{M}_D(j,\, j) \quad \text{with } 1 \leqslant i < j \leqslant N, \tag{23}$$

then the system has no identical pairs of characteristic roots. In all such cases, the system is not defective and it is found that

$$\underline{\mathbf{U}_L^{\mathrm{T}}}\underline{\mathbf{K}}\underline{\mathbf{U}_R} = \underline{\mathbf{K}_D}, \qquad \underline{\mathbf{U}_L^{\mathrm{T}}}\underline{\mathbf{D}}\underline{\mathbf{U}_R} = \underline{\mathbf{D}_D}, \qquad \underline{\mathbf{U}_L^{\mathrm{T}}}\underline{\mathbf{M}}\underline{\mathbf{U}_R} = \underline{\mathbf{M}_D}. \tag{24}$$

A proof of this theorem is given in Appendix A based on the development of $\{\underline{\mathbf{U}_L}, \underline{\mathbf{U}_R}\}$ from the left and right matrices of complex modes respectively.

Equations (24) are a concise expression of the fact that for almost every second order system comprising the real $(N \times N)$ matrices $\{\mathbf{K}, \mathbf{D}, \mathbf{M}\}$, there is a real diagonalizing transformation in the form of the real $(2N \times 2N)$ matrices $\{\underline{\mathbf{U}_L}, \underline{\mathbf{U}_R}\}$ which maps this onto a diagonal second order system comprising $\{\mathbf{K}_D, \mathbf{D}_D, \mathbf{M}_D\}$.

The development of equations (24) from a conventional complex formulation is summarized in Appendix A for convenience so that the relationship between $\{\underline{\mathbf{U}_L}, \underline{\mathbf{U}_R}\}$ and the more familiar complex modal matrices can be understood. If any two of the equations in equations (24) are satisfied and if $\{\underline{\mathbf{U}_L}, \underline{\mathbf{U}_R}\}$ are invertible (which they must be if the system is not defective), then it is straightforward to show the third equation must also be satisfied. This follows immediately from the observations that if $\mathbf{K}$ is invertible,

$$\underline{\mathbf{D}\mathbf{K}}^{-1}\underline{\mathbf{D}} \equiv \underline{\mathbf{M}} \tag{25}$$

and if $\mathbf{M}$ is invertible

$$\underline{\mathbf{D}\mathbf{M}}^{-1}\underline{\mathbf{D}} \equiv \underline{\mathbf{K}}. \tag{26}$$

Cases where $\mathbf{K}$ is not invertible can be addressed directly by replacing $\mathbf{K}$ with $(\mathbf{K} + \varepsilon\Delta\mathbf{K})$ where $\Delta\mathbf{K}$ is any matrix chosen such that $(\mathbf{K} + \varepsilon\Delta\mathbf{K})$ is non-singular for any positive real scalar, ε, smaller than some limiting value and taking the limit as $\varepsilon \to 0$. Similarly for cases in which $\mathbf{M}$ is singular. An alternative is to select some real eigenvalue shift, α, in the eigenvalues such that the system having the shifted eigenvalues is represented as $\{(\mathbf{K}+\alpha\mathbf{D} + \alpha^2\mathbf{M}), (\mathbf{D} + 2\alpha\mathbf{M}), (\mathbf{M})\}$ instead of $\{(\mathbf{K}, \mathbf{D}, \mathbf{M}\}$.

Although the link with the complex modes is made in Appendix A, this paper treats equations (24) as the fundamental definition of characteristic behaviour for real second order systems. Partition $\{\underline{\mathbf{U}_L}, \underline{\mathbf{U}_R}\}$ as follows

$$\underline{U_R} =: \begin{bmatrix} \mathbf{W}_{RD} & \mathbf{X}_{RD} \\ \mathbf{Y}_{RD} & \mathbf{Z}_{RD} \end{bmatrix}, \qquad \underline{U_L} =: \begin{bmatrix} \mathbf{W}_{LD} & \mathbf{X}_{LD} \\ \mathbf{Y}_{LD} & \mathbf{Z}_{LD} \end{bmatrix}. \tag{27}$$

Then, the following features motivate the acceptance of equation (24) as the fundamental definition of characteristic behaviour for second order systems:

- Systems comprising real matrices $\{\mathbf{K}, \mathbf{D}, \mathbf{M}\}$ produce real outcomes, $\{\underline{\mathbf{U}_L}, \underline{\mathbf{U}_R}\}$ and $\{\mathbf{K}_D, \mathbf{D}_D, \mathbf{M}_D\}$ having direct physical interpretations.
- The diagonalizing transformation is well defined for systems in which either (or both) of $\{\mathbf{K}_D, \mathbf{M}_D\}$ are singular. In contrast, none of the possible definitions of complex roots and associated eigenvectors can deal with the case where both are singular. Some formulations can deal with the case where $\mathbf{K}_D$ is singular but not $\mathbf{M}_D$ and others with the converse case. This point is clear when one considers the case

$$\left\{\mathbf{K} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{M} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}.$$

- There is no fundamental distinction between "real roots" and "complex roots" in the diagonalizing transformation. The rates of change of $\{\underline{\mathbf{U}_L}, \underline{\mathbf{U}_R}\}$ and $\{\mathbf{K}_D, \mathbf{D}_D, \mathbf{M}_D\}$ are all well defined with respect to any parameter which causes a pair of complex roots to transform into a pair of real roots. The derivatives of a pair of complex modes with respect to any such parameter are undefined at the instant of change between complex and real modes.
- For classically damped systems, the mass-normalized modes of the undamped system appear in $\{\mathbf{W}_{LD}, \mathbf{W}_{RD}, \mathbf{Z}_{LD}, \mathbf{Z}_{RD}\}$ (with $\mathbf{W}_{LD} = \mathbf{Z}_{RD}$ and $\mathbf{W}_{LD} = \mathbf{Z}_{RD}$) and matrices $\{\mathbf{X}_{LD}, \mathbf{X}_{RD}, \mathbf{Y}_{LD}, \mathbf{Y}_{RD}\}$ contain only zeros.
- For self-adjoint systems (systems characterized by symmetric $\{\mathbf{K}, \mathbf{D}, \mathbf{M}\}$) an appropriate choice of scaling leads to $\mathbf{W}_{LD} = \mathbf{W}_{RD}, \mathbf{X}_{LD} = \mathbf{X}_{RD}, \mathbf{Y}_{LD} = \mathbf{Y}_{RD}$ and $\mathbf{Z}_{LD} = \mathbf{Z}_{RD}$.
- For conservative systems characterized by symmetric $\{\mathbf{K},\mathbf{M}\}$ and skew-symmetric $\mathbf{D}$, an appropriate choice of scaling leads to $\mathbf{W}_{LD} = \mathbf{Z}_{RD}$ $\mathbf{X}_{LD} = -\mathbf{X}_{RD}$, $Y_{\mathrm{LD}} = -Y_{\mathrm{RD}}$ and $\mathbf{Z}_{LD} = \mathbf{Z}_{RD}$.
- The structure of $\underline{\mathbf{K}_D}$ is identical to that of $\underline{\mathbf{K}}$. Similarly, the structure of $\underline{\mathbf{D}_D}$ is identical to that of $\underline{\mathbf{D}}$ and the structure of $\underline{\mathbf{M}_D}$ is identical to that of $\underline{\mathbf{M}}$.

When $\mathbf{M}$ is full rank, it is always possible to scale $\{\underline{\mathbf{U}_L}, \underline{\mathbf{U}_R}\}$ by real scalars such that $\mathbf{M}_D = \mathbf{I}$. For symmetric $\{\mathbf{K}, \mathbf{D}, \mathbf{M}\}$, simultaneously achieving $\underline{\mathbf{U}_L} = \underline{\mathbf{U}_R}$ and $\mathbf{M}_D = \mathbf{I}$ is possible only if $\mathbf{M}$ is positive definite.

A simple two-degree-of-freedom example is provided here. Suppose the mass, damping and stiffness matrices are given by

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 0{\cdot}4 & -0{\cdot}4 \\ -0{\cdot}4 & 0{\cdot}4 \end{bmatrix}, \qquad \mathbf{K} = \begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix}. \tag{28}$$

The transformed diagonal matrices are

$$\mathbf{K}_D = \mathrm{diag}\begin{bmatrix} 1{\cdot}752 & 5{\cdot}138 \end{bmatrix}, \quad \mathbf{D}_D = \mathrm{diag}\begin{bmatrix} 0{\cdot}1768 & 0{\cdot}6232 \end{bmatrix}, \quad \mathbf{M}_D = \mathrm{diag}\begin{bmatrix} 1{\cdot}0 & 1{\cdot}0 \end{bmatrix} \tag{29}$$

and the transformation is given by

$$\mathbf{W}_{LD} = \mathbf{W}_{RD} = \begin{bmatrix} 0{\cdot}9731 & -0{\cdot}3541 \\ 0{\cdot}3171 & 0{\cdot}9451 \end{bmatrix}, \qquad \mathbf{X}_{LD} = \mathbf{X}_{RD} = \begin{bmatrix} -0{\cdot}03079 & -0{\cdot}09237 \\ 0{\cdot}09033 & -0{\cdot}02926 \end{bmatrix}, \tag{30}$$

$$\mathbf{Y}_{LD} = \mathbf{Y}_{RD} = \begin{bmatrix} -0{\cdot}05394 & 0{\cdot}4746 \\ -0{\cdot}1582 & 0{\cdot}1503 \end{bmatrix}, \qquad \mathbf{Z}_{LD} = \mathbf{Z}_{RD} = \begin{bmatrix} 0{\cdot}9786 & -0{\cdot}2965 \\ 0{\cdot}3011 & 0{\cdot}9633 \end{bmatrix}.$$

3. GENERAL CO-ORDINATE TRANSFORMATIONS FOR SECOND ORDER SYSTEMS AND STRUCTURE-PRESERVING TRANSFORMATIONS

In equations (24), $\{\underline{\mathbf{U}_L}, \underline{\mathbf{U}_R}\}$ have special significance as the transformation taking the original system $\{\mathbf{K}, \mathbf{D}, \mathbf{M}\}$ into diagonal form $\{\mathbf{K}_D, \mathbf{D}_D, \mathbf{M}_D\}$. In this section equations (24) are used to show that any pair of $(2N \times 2N)$ matrices, $\{\underline{\mathbf{T}_L}, \underline{\mathbf{T}_R}\}$, implicitly defines a co-ordinate transformation for an N-degree-of-freedom second order system. The special subset of *structure-preserving transformations* is also defined here.

Any pair of $(N \times N)$ matrices, $\{\mathbf{T}_L, \mathbf{T}_R\}$ can be said to define a co-ordinate transformation according to

$$\mathbf{q} = \mathbf{T}_R\mathbf{r}, \qquad \mathbf{R} = \mathbf{T}_L^{\mathrm{T}}\mathbf{Q}, \qquad \mathbf{K}_{rr}\mathbf{r} + \mathbf{D}_{rr}\dot{\mathbf{r}} + \mathbf{M}_{rr}\ddot{\mathbf{r}} = \mathbf{R},$$

with

$$\mathbf{K}_{rr} = \mathbf{T}_L^{\mathrm{T}}\mathbf{K}\mathbf{T}_R, \qquad \mathbf{D}_{rr} = \mathbf{T}_L^{\mathrm{T}}\mathbf{D}\mathbf{T}_R, \qquad \mathbf{M}_{rr} = \mathbf{T}_L^{\mathrm{T}}\mathbf{M}\mathbf{T}_R. \tag{31}$$

Equations (31) comprise the general case of what is normally considered (in the structural dynamics community) to be a co-ordinate transformation. Co-ordinate transformations in the form of equations (31) will be described as *first order* transformations here since they are the only transformations which would ever be applied to a first order system ($\mathbf{M} = \mathbf{0}$). The transformation of equation (12) (which can diagonalize any classically damped system) is a special case of a *first order* co-ordinate transformation.

Provided that $\{\mathbf{T}_L, \mathbf{T}_R\}$ are both square and full rank, they may be chosen arbitrarily and the same response will be computed for the system using either the original or the transformed representation. The same characteristic behaviour will also be obtained in both cases. If $\mathbf{T}_L$ and $\mathbf{T}_R$ have fewer columns than rows, then the co-ordinate transformation implicitly imposes constraints and reduces the number of system degrees of freedom. In this case, both the response and the characteristic behaviour are modified in general and the transformation is a *model-reducing* transformation.

The space of co-ordinate transformations for second order systems includes the full space of all *first order* co-ordinate transformations as a subspace. Suppose, now, that one pair of $(2N \times 2N)$ matrices, $\{\underline{\mathbf{T}_{L1}}, \underline{\mathbf{T}_{R1}}\}$, is selected arbitrarily and that another pair of $(2N \times 2N)$ matrices, $\{\underline{\mathbf{T}_{L2}}, \underline{\mathbf{T}_{R2}}\}$ is computed to satisfy

$$\underline{\mathbf{T}_{L1}\mathbf{T}_{L2}} = \underline{\mathbf{U}_L}, \qquad \underline{\mathbf{T}_{R1}\mathbf{T}_{R2}} = \underline{\mathbf{U}_R}. \tag{32}$$

Substitute these into equations (24) to obtain

$$\underline{\mathbf{T}_{L2}^{\mathrm{T}}}\left(\underline{\mathbf{T}_{L1}^{\mathrm{T}}}\underline{\mathbf{K}}\underline{\mathbf{T}_{R1}}\right)\underline{\mathbf{T}_{R2}} = \underline{\mathbf{K}_D}, \qquad \underline{\mathbf{T}_{L2}^{\mathrm{T}}}\left(\underline{\mathbf{T}_{L1}^{\mathrm{T}}}\underline{\mathbf{D}}\underline{\mathbf{T}_{R1}}\right)\underline{\mathbf{T}_{R2}} = \underline{\mathbf{D}_D},$$
$$\underline{\mathbf{T}_{L2}^{\mathrm{T}}}\left(\underline{\mathbf{T}_{L1}^{\mathrm{T}}}\underline{\mathbf{M}}\underline{\mathbf{T}_{R1}}\right)\underline{\mathbf{T}_{R2}} = \underline{\mathbf{M}_D}. \tag{33}$$

Evidently, a transformation has been carried out having the following effects:

$$\underline{\mathbf{K}} \Rightarrow \left(\underline{\mathbf{T}_{L1}^{\mathrm{T}}}\underline{\mathbf{K}}\underline{\mathbf{T}_{R1}}\right) =: \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}, \tag{34}$$

$$\underline{\mathbf{D}} \Rightarrow \left(\underline{\mathbf{T}_{L1}^{\mathrm{T}}}\underline{\mathbf{D}}\underline{\mathbf{T}_{R1}}\right) =: \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix}, \tag{35}$$

$$\underline{\mathbf{M}} \Rightarrow \left(\underline{\mathbf{T}_{L1}^{\mathrm{T}}}\underline{\mathbf{M}}\underline{\mathbf{T}_{R1}}\right) =: \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix} \tag{36}$$

and matrices $\{\underline{\mathbf{T}_{L2}}, \underline{\mathbf{T}_{R2}}\}$ now assume the *diagonalizing transformation* role formerly played by $\{\underline{\mathbf{U}_L}, \underline{\mathbf{U}_R}\}$.

If $\{\underline{\mathbf{T}_{L1}}, \underline{\mathbf{T}_{R1}}\}$ are chosen completely arbitrarily, the transformed representation of the system will lack some of the structure possessed by the original representation. The primary focus of this paper is on *structure-preserving* similarity transformations for second order systems.

Transformation $\{\underline{\mathbf{T}_{L1}}, \underline{\mathbf{T}_{R1}}\}$ is defined as a *structure-preserving transformation* if and only if

$$\mathbf{A}_{11} = \mathbf{0}, \qquad \mathbf{B}_{12} = \mathbf{0}, \qquad \mathbf{B}_{21} = \mathbf{0}, \qquad \mathbf{C}_{22} = \mathbf{0}, \tag{37}$$

$$\mathbf{A}_{12} = \mathbf{B}_{11} = \mathbf{A}_{21}, \qquad \mathbf{C}_{12} = -\mathbf{B}_{22} = \mathbf{C}_{21} \qquad \mathbf{A}_{22} = \mathbf{C}_{11}. \tag{38−40}$$

In such cases, the original system $\{\mathbf{K}, \mathbf{D}, \mathbf{M}\}$ is transformed into a new system $\{\mathbf{K}', \mathbf{D}', \mathbf{M}'\}$ having the same characteristic roots as the original system. Note that not all of these conditions are independent. Some are automatically satisfied as a direct result of equations (25) and (26).

4. TIME-DOMAIN RESPONSE USING THE DIAGONALIZING TRANSFORMATION

Equations (15) and (16) are *first order* state-space representations insofar as they each involve the zeroth and first derivatives of the state vector. Both of these equations are commonly encountered in the analysis of generally damped systems. Equation (17) is a *second order* state-space representation since it involves the zeroth and second derivatives of the state-vector only. The definition of $\mathbf{p}$ is implicit in all three of equations (15)–(17) and the definition of $\mathbf{P}$ is implicitly contained also in equation (17). Any one of equations (15)–(17) is normally adequate to describe the time-domain behaviour of the system.

Consider that the following co-ordinate transformation is carried out based on matrices $\{\mathbf{W}_{RD}, \mathbf{X}_{RD}, \mathbf{Y}_{RD}, \mathbf{Z}_{RD}\}$ (which comprise $\underline{\mathbf{U}_R}$ (cf., equation (27)))

$$\begin{bmatrix} \mathbf{q} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{W}_{RD} & \mathbf{X}_{RD} \\ \mathbf{Y}_{RD} & \mathbf{Z}_{RD} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{u} \end{bmatrix}. \tag{41}$$

Together, vectors $\mathbf{u}$ and $\mathbf{v}$ represent the state of the system unambiguously. Note that if all of the original displacement co-ordinates are translations, the (SI) units for $\mathbf{v}$ are (m) and the (SI) units for $\mathbf{u}$ are (m/s)—in perfect consistency with the units for $\mathbf{q}$ and $\mathbf{p}$ respectively. Define new excitation vectors, $\mathbf{U}$ and $\mathbf{V}$, in terms of the force vector, $\mathbf{Q}$, and its first derivative, $\mathbf{P}$ using $\{\mathbf{W}_{LD}, \mathbf{X}_{LD}, \mathbf{Y}_{LD}, \mathbf{Z}_{LD}\}$ as

$$\begin{bmatrix} \mathbf{U} \\ \mathbf{V} \end{bmatrix} = \begin{bmatrix} \mathbf{W}_{LD} & \mathbf{X}_{LD} \\ \mathbf{Y}_{LD} & \mathbf{Z}_{LD} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{P} \\ \mathbf{Q} \end{bmatrix}. \tag{42}$$

Substitute for the state vector in each of equations (15)–(17) using equation (41) and pre-multiply each of these three equations by the $\underline{\mathbf{U}_L}^T$. Provided that equations (24) are satisfied, it is evident that the transformed equations are

$$\begin{bmatrix} \mathbf{0} & \mathbf{K}_D \\ \mathbf{K}_D & \mathbf{D}_D \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{u} \end{bmatrix} - \begin{bmatrix} \mathbf{K}_D & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_D \end{bmatrix} \begin{bmatrix} \dot{\mathbf{v}} \\ \dot{\mathbf{u}} \end{bmatrix} = \begin{bmatrix} \mathbf{Y}_{LD}^{\mathrm{T}} \\ \mathbf{Z}_{LD}^{\mathrm{T}} \end{bmatrix} \mathbf{Q}, \tag{43}$$

$$\begin{bmatrix} \mathbf{K}_D & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_D \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{u} \end{bmatrix} + \begin{bmatrix} \mathbf{D}_D & \mathbf{M}_D \\ \mathbf{M}_D & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{v}} \\ \dot{\mathbf{u}} \end{bmatrix} = \begin{bmatrix} \mathbf{W}_{LD}^{\mathrm{T}} \\ \mathbf{X}_{LD}^{\mathrm{T}} \end{bmatrix} \mathbf{Q}, \tag{44}$$

$$\begin{bmatrix} \mathbf{0} & \mathbf{K}_D \\ \mathbf{K}_D & \mathbf{D}_D \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{u} \end{bmatrix} + \begin{bmatrix} \mathbf{D}_D & \mathbf{M}_D \\ \mathbf{M}_D & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{v}} \\ \ddot{\mathbf{u}} \end{bmatrix} = \begin{bmatrix} \mathbf{U} \\ \mathbf{V} \end{bmatrix}. \tag{45}$$

Taking equation (45) in conjunction with equations (41) and (42) yields a *diagonalized second order state-space* representation of the original system. Comparing equation (45) with equation (17) makes it clear that response can be computed as the sum of responses from N decoupled systems.

If $\mathbf{U}$ was identical to the first derivative of $\mathbf{V}$ with respect to time, there could be no hesitation in identifying these decoupled systems as single-degree-of-freedom second order systems. In general, $\mathbf{U}$ is not identical to the first derivative of $\mathbf{V}$ with respect to time. In fact, this identity holds true only for classically damped systems. Correspondingly (and more importantly) $\mathbf{v}$ is not equal to the first derivative of $\mathbf{u}$ with respect to time when forcing is present on the system but when the system is in free vibration ($\mathbf{Q}=0$), this identity holds as can be seen from equation (43) or (44).

To support the assertion that there is a real co-ordinate transformation mapping (almost) any general second order system onto a second order system characterized by diagonal matrices, it is necessary only to demonstrate that there is some second order system described by

$$\mathbf{K}_D\mathbf{z} + \mathbf{D}_D\dot{\mathbf{z}} + \mathbf{M}_D\ddot{\mathbf{z}} = \mathbf{Z} \tag{46}$$

and some set of relationships giving $\mathbf{Z}(t)$ in terms of $\mathbf{Q}(t)$ and $\mathbf{q}(t)$ in terms of $\mathbf{z}(t)$ such that $\mathbf{q}(t)$ may be computed from $\mathbf{Q}(t)$ as the sum of N individual contributions using equation (41). Expanding equation (43) leads to

$$\mathbf{K}_D(\mathbf{u} - \dot{\mathbf{v}}) = \mathbf{Y}_{LD}^{\mathrm{T}}\mathbf{Q}, \tag{47}$$

$$\mathbf{M}_D\dot{\mathbf{u}} + \mathbf{D}_D\mathbf{u} + \mathbf{K}_D\mathbf{v} = \mathbf{Z}_{LD}^{\mathrm{T}}\mathbf{Q}. \tag{48}$$

Differentiate equation (48), substitute for $\mathbf{v}$ using equation (47) and apply the definition of $\mathbf{P}$ in equation (14) to obtain

$$\mathbf{M}_D\ddot{\mathbf{u}} + \mathbf{D}_D\dot{\mathbf{u}} + \mathbf{K}_D\mathbf{u} = (\mathbf{Y}_{LD}^{\mathrm{T}}\mathbf{Q} + \mathbf{Z}_{LD}^{\mathrm{T}}\mathbf{P}). \tag{49}$$

This is in precisely the form of equation (46). The forcing term on the right-hand side of equation (49), is computed as a linear combination of $\mathbf{Q}(t)$ and $\mathbf{P}(t)$. To recover the response, $\mathbf{q}(t)$, it is first necessary to generate $\mathbf{v}(t)$ from $\mathbf{u}(t)$. This can be achieved directly using equation (48) if the first derivative of $\mathbf{u}(t)$ has been stored together with $\mathbf{u}(t)$. Equation (41) can then be employed to recover $\mathbf{q}(t)$ and $\mathbf{p}(t)$.

5. TIME-DOMAIN RESPONSE UNDER FULLY GENERAL TRANSFORMATIONS

Consider again that ($N \times N$) matrices, $\{\mathbf{W}_{L1}, \mathbf{X}_{L1}, \mathbf{Y}_{L1}, \mathbf{Z}_{L1}\}$ and $\{\mathbf{W}_{R1}, \mathbf{X}_{R1}, \mathbf{Y}_{R1}, \mathbf{Z}_{R1}\}$, are arbitrarily selected and that $\{\mathbf{A}_{11}, \mathbf{A}_{12}, \mathbf{A}_{21}, \mathbf{A}_{22}\}$, $\{\mathbf{B}_{11}, \mathbf{B}_{12}, \mathbf{B}_{21}, \mathbf{B}_{22}\}$ and $\{\mathbf{C}_{11}, \mathbf{C}_{12}, \mathbf{C}_{21}, \mathbf{C}_{22}\}$ are then computed according to equations (34)–(36). Apply the transformation as follows:

$$\begin{bmatrix} \mathbf{q} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{W}_{R1} & \mathbf{X}_{R1} \\ \mathbf{Y}_{R1} & \mathbf{Z}_{R1} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{u} \end{bmatrix} =: \underline{\mathbf{T}_{R1}} \begin{bmatrix} \mathbf{v} \\ \mathbf{u} \end{bmatrix}, \qquad \begin{bmatrix} \mathbf{U} \\ \mathbf{V} \end{bmatrix} = \begin{bmatrix} \mathbf{W}_{L1} & \mathbf{X}_{L1} \\ \mathbf{Y}_{L1} & \mathbf{Z}_{L1} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{P} \\ \mathbf{Q} \end{bmatrix} =: \underline{\mathbf{T}_{L1}^{\mathrm{T}}} \begin{bmatrix} \mathbf{P} \\ \mathbf{Q} \end{bmatrix}. \tag{50}$$

Vectors $\{\mathbf{u}, \mathbf{v}, \mathbf{U}, \mathbf{V}\}$ here represent different quantities from those of equations (41) and (42). Evidently, the system response can be computed using any one of the following three equations obtained from equations (15)–(17) by applying equations (50)

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{u} \end{bmatrix} - \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{v}} \\ \dot{\mathbf{u}} \end{bmatrix} = \begin{bmatrix} \mathbf{Y}_{L1}^{\mathrm{T}} \\ \mathbf{Z}_{L1}^{\mathrm{T}} \end{bmatrix} \mathbf{Q}, \tag{51}$$

$$\begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{u} \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{v}} \\ \dot{\mathbf{u}} \end{bmatrix} = \begin{bmatrix} \mathbf{W}_{L1}^{\mathrm{T}} \\ \mathbf{X}_{L1}^{\mathrm{T}} \end{bmatrix} \mathbf{Q}, \tag{52}$$

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{u} \end{bmatrix} - \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{v}} \\ \ddot{\mathbf{u}} \end{bmatrix} = \begin{bmatrix} \mathbf{U} \\ \mathbf{V} \end{bmatrix}. \tag{53}$$

To compute the time-domain response, it is only strictly necessary to be able to compute the second derivative of $\mathbf{q}$ given its zeroth and first derivatives. It is obviously advantageous to compute the third derivative also since this provides a numerical integration process with considerable additional information. Equations (17), (45) and (53) all provide for the direct computation of this derivative but all three of these require $\mathbf{P}$ — the first derivative of force with respect to time. Any errors in the estimation of $\mathbf{P}(=\dot{\mathbf{Q}})$ will affect only the calculated third derivative of displacement with respect to time.

6. FREQUENCY-DOMAIN RESPONSE COMPUTATIONS

The ability to compute the steady state response of a system to harmonic forcing is one of the key functions of any model. Conventionally, a complex dynamic stiffness matrix is formed, forces are represented by a complex vector and response is computed as another complex vector. Equation (17) provides what is arguably a more direct approach.

Discount, initially, that in the steady state, $\mathbf{p}(=\dot{\mathbf{q}})$ has a known magnitude and phase relationship to $\mathbf{q}$ and that $\mathbf{P}(=\dot{\mathbf{Q}})$ has a known magnitude and phase relationship to $\mathbf{Q}$. Using only the knowledge that all of these quantities vary sinusoidally with respect to time

$$\begin{bmatrix} \mathbf{q} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{q}_{\cos} & \mathbf{q}_{\sin} \\ \mathbf{p}_{\cos} & \mathbf{p}_{\sin} \end{bmatrix} \begin{bmatrix} \cos(\omega t) \\ \sin(\omega t) \end{bmatrix}, \qquad \begin{bmatrix} \mathbf{P} \\ \mathbf{Q} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{\cos} & \mathbf{P}_{\sin} \\ \mathbf{Q}_{\cos} & \mathbf{Q}_{\sin} \end{bmatrix} \begin{bmatrix} \cos(\omega t) \\ \sin(\omega t) \end{bmatrix}. \tag{54}$$

Recognizing that differentiating a sinusoidal function twice returns $-\omega^2$ times the same function, the following is obtained from equation (17):

$$\left[\begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K} & \mathbf{D} \end{bmatrix} - \omega^2 \begin{bmatrix} \mathbf{D} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \right] \begin{bmatrix} \mathbf{q}_{\cos} & \mathbf{q}_{\sin} \\ \mathbf{p}_{\cos} & \mathbf{p}_{\sin} \end{bmatrix} \begin{bmatrix} \cos(\omega t) \\ \sin(\omega t) \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{\cos} & \mathbf{P}_{\sin} \\ \mathbf{Q}_{\cos} & \mathbf{Q}_{\sin} \end{bmatrix} \begin{bmatrix} \cos(\omega t) \\ \sin(\omega t) \end{bmatrix}. \tag{55}$$

Because equation (55) must be true for all time, t, the time dependency can be removed. Now, incorporate the known relationships

$$\begin{aligned} \mathbf{P}_{\cos} &= -\omega \mathbf{Q}_{\sin}, & \mathbf{P}_{\sin} &= \omega \mathbf{Q}_{\cos}, \\ \mathbf{p}_{\cos} &= -\omega \mathbf{q}_{\sin}, & \mathbf{p}_{\sin} &= \omega \mathbf{q}_{\cos} \end{aligned} \tag{56}$$

to find

$$\left[\begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K} & \mathbf{D} \end{bmatrix} - \omega^2 \begin{bmatrix} \mathbf{D} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \right] \begin{bmatrix} \mathbf{I}_{(N\times N)} & \mathbf{0} \\ \mathbf{0} & \omega \mathbf{I}_{(N\times N)} \end{bmatrix} \begin{bmatrix} \mathbf{q}_{\cos} & \mathbf{q}_{\sin} \\ -\mathbf{q}_{\sin} & \mathbf{q}_{\cos} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \omega \mathbf{I}_{(N\times N)} \\ \mathbf{I}_{(N\times N)} & \mathbf{0} \end{bmatrix}$$
$$\times \begin{bmatrix} \mathbf{Q}_{\cos} & \mathbf{Q}_{\sin} \\ -\mathbf{Q}_{\sin} & \mathbf{Q}_{\cos} \end{bmatrix}. \tag{57}$$

With some trivial simplification

$$\begin{bmatrix} (\mathbf{K}-\omega^2\mathbf{M}) & \omega\mathbf{D} \\ -\omega\mathbf{D} & (\mathbf{K}-\omega^2\mathbf{M}) \end{bmatrix} \begin{bmatrix} \mathbf{q}_{\cos} & \mathbf{q}_{\sin} \\ -\mathbf{q}_{\sin} & \mathbf{q}_{\cos} \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_{\cos} & \mathbf{Q}_{\sin} \\ -\mathbf{Q}_{\sin} & \mathbf{Q}_{\cos} \end{bmatrix}. \tag{58}$$

Equation (58) can be expressed compactly using the algebra of complex numbers and this is a computationally sensible way to solve this equation directly.

If the diagonalizing transformation is applied to the system before computing frequency response, it is found that at each value of ω, the vector coefficients, $\{\mathbf{u}_{\cos}, \mathbf{v}_{\cos}, \mathbf{u}_{\sin}, \mathbf{v}_{\sin}\}$ of $\cos(\omega t)$ and $\sin(\omega t)$ must be determined for vectors $\mathbf{u}$ and $\mathbf{v}$ as defined in equation (41). Complex numbers are not useful in this case since $\mathbf{u} \neq \dot{\mathbf{v}}$ and, as a result, the system of equations to be solved does not have the structure of equation (58). However, the system of equations does comprise N decoupled pairs of equations and for this reason, solution is computationally very efficient:

$$\begin{bmatrix} \mathbf{q}_{\cos} & \mathbf{q}_{\sin} \\ \mathbf{p}_{\cos} & \mathbf{p}_{\sin} \end{bmatrix} = \underline{\mathbf{U}_R}\left[\underline{\mathbf{K}_D} - \omega^2\underline{\mathbf{M}_D}\right]^{-1}\underline{\mathbf{U}_L^{\mathrm{T}}} \begin{bmatrix} \mathbf{P}_{\cos} & \mathbf{P}_{\sin} \\ \mathbf{Q}_{\cos} & \mathbf{Q}_{\sin} \end{bmatrix}. \tag{59}$$

As expected, the diagonalizing transformation enables frequency response to be composed as the superposition of contributions from individual single-degree-of-freedom systems.

7. MODEL-REDUCING STRUCTURE-PRESERVING TRANSFORMATIONS

All of the transformations dealt with up to this point have been square and full rank. The transformed models in all cases are perfectly equivalent to the original models provided that the transformation matrices are used correctly to map from physical force sets into the new force sets and then to map from the response computed in the new co-ordinates back to physical responses. There is no obvious role for rank-deficient transformations but transformations that are rectangular are very common and very useful—especially for the purposes of model reduction. These are considered briefly here.

First order co-ordinate transformations which are model reducing are generated by selecting $(N \times M)$ transformation matrices, $\mathbf{T}_R$ and $\mathbf{T}_L$, in equation (31) with $M < N$. There is a substantial literature on these first order model-reducing transformations [25–28].

By extension, it is possible to generate arbitrary $(2N \times 2\,M)$ matrices, $\left\{\underline{\mathbf{T}_{L1}}, \underline{\mathbf{T}_{R1}}\right\}$, with $M < N$ and using these, a new reduced-dimension representation of the second order system can be generated in the form of a pair of equations comprising equation (50) and any one of equations (51)–(53) where the definitions of equations (34)–(36) are applied.

Model-reducing transformations may be *structure-preserving* just as square transformations may be. The transformation represented by $\{\mathbf{W}_{L1}, \mathbf{X}_{L1}, \mathbf{Y}_{L1}, \mathbf{Z}_{L1}\}$ and $\{\mathbf{W}_{R1}, \mathbf{X}_{R1}, \mathbf{Y}_{R1}, \mathbf{Z}_{R1}\}$ is a *structure-preserving* model-reducing transformation for the system $\{\mathbf{K}, \mathbf{D}, \mathbf{M}\}$ if there is some new system represented by the $(M \times M)$ matrices $\{\mathbf{K}', \mathbf{D}', \mathbf{M}'\}$ such that

$$\begin{bmatrix} \mathbf{W}_{L1} & \mathbf{X}_{L1} \\ \mathbf{Y}_{L1} & \mathbf{Z}_{L1} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{W}_{R1} & \mathbf{Y}_{R1} \\ \mathbf{Y}_{R1} & \mathbf{Z}_{R1} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{K}' \\ \mathbf{K}' & \mathbf{D}' \end{bmatrix}, \tag{60}$$

$$\begin{bmatrix} \mathbf{W}_{L1} & \mathbf{X}_{L1} \\ \mathbf{Y}_{L1} & \mathbf{Z}_{L1} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{W}_{R1} & \mathbf{Y}_{R1} \\ \mathbf{Y}_{R1} & \mathbf{Z}_{R1} \end{bmatrix} = \begin{bmatrix} \mathbf{K}' & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}' \end{bmatrix}, \tag{61}$$

$$\begin{bmatrix} \mathbf{W}_{L1} & \mathbf{X}_{L1} \\ \mathbf{Y}_{L1} & \mathbf{Z}_{L1} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{D} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{W}_{R1} & \mathbf{Y}_{R1} \\ \mathbf{Y}_{R1} & \mathbf{Z}_{R1} \end{bmatrix} = \begin{bmatrix} \mathbf{D}' & \mathbf{M}' \\ \mathbf{M}' & \mathbf{0} \end{bmatrix}. \tag{62}$$

In the case of square (full rank) structure-preserving transformations, any two equations from equations (60)–(62) would be sufficient to ensure that the third equation was automatically satisfied. This is not the case for model-reducing transformations. One particular model-reducing transformation is introduced in reference [24] as an extension of static reduction [25] to the context of generally damped systems. This transformation satisfies equations (60) and (61) but it does not also satisfy equation (62) and strictly, therefore, it is not structure preserving.

8. EXAMPLE

A 4-degree of freedom system is considered. The stiffness and mass matrices for this system are diagonal and the damping matrix is fully populated. These matrices are,

$$\mathbf{K} = \mathrm{diag}[0{\cdot}7 \quad 1 \quad 4 \quad 9], \quad \mathbf{M} = \mathrm{diag}[1 \quad 1 \quad 1 \quad 1], \quad \mathbf{D} = \begin{bmatrix} 0{\cdot}9 & -0{\cdot}5 & 0{\cdot}4 & 0{\cdot}2 \\ -0{\cdot}5 & 0{\cdot}8 & -0{\cdot}8 & -0{\cdot}2 \\ 0{\cdot}4 & -0{\cdot}8 & 1{\cdot}0 & 0{\cdot}2 \\ 0{\cdot}2 & -0{\cdot}2 & 0{\cdot}2 & 0{\cdot}6 \end{bmatrix}. \tag{63}$$

In this example, three different transformations are given. The purpose of the example is simply to illustrate that these transformations do exist for general systems. There is no loss of generality in the fact that the example begins with diagonal **K** and **M** since it is trivial to transform any general system into this form.

8.1. TRANSFORMATION TO DIAGONAL FORM

The transformation to diagonal form as represented by equations (15)–(17) results in the following diagonal system $\{\mathbf{K}_D, \mathbf{D}_D, \mathbf{M}_D\}$:

$$[\mathrm{diag}(\mathbf{K}_D) \qquad \mathrm{diag}(\mathbf{D}_D) \qquad \mathrm{diag}(\mathbf{M}_D)]$$
$$= \begin{bmatrix} 9{\cdot}0296807\mathrm{E}-001 & 4{\cdot}1698283\mathrm{E}-001 & 1{\cdot}0000000\mathrm{E}+000 \\ 9{\cdot}9076461\mathrm{E}-001 & 1{\cdot}6526387\mathrm{E}+000 & 1{\cdot}0000000\mathrm{E}+000 \\ 3{\cdot}1653441\mathrm{E}+000 & 6{\cdot}5971716\mathrm{E}-001 & 1{\cdot}0000000\mathrm{E}+000 \\ 8{\cdot}8989084\mathrm{E}+000 & 5{\cdot}7066133\mathrm{E}-001 & 1{\cdot}0000000\mathrm{E}+000 \end{bmatrix}. \tag{64}$$

As the system matrices are symmetrical, the left and right transformation matrices are identical and these are as follows:

$$\mathbf{W}_{LD} = \mathbf{W}_{RD}$$
$$= \begin{bmatrix} 6{\cdot}6491295\mathrm{E}-001 & 1{\cdot}0006184\mathrm{E}+000 & -2{\cdot}2141319\mathrm{E}-001 & -2{\cdot}5219434\mathrm{E}-002 \\ 8{\cdot}1094897\mathrm{E}-001 & -6{\cdot}1504382\mathrm{E}-001 & 1{\cdot}6717115\mathrm{E}-001 & 3{\cdot}6003530\mathrm{E}-002 \\ -4{\cdot}8738704\mathrm{E}-002 & 4{\cdot}2683032\mathrm{E}-002 & 9{\cdot}8786480\mathrm{E}-001 & -5{\cdot}3803347\mathrm{E}-002 \\ -6{\cdot}3093553\mathrm{E}-003 & 1{\cdot}9096531\mathrm{E}-003 & 4{\cdot}3051685\mathrm{E}-002 & 9{\cdot}9856527\mathrm{E}-001 \end{bmatrix}, \tag{65}$$

$$\mathbf{X}_{LD} = \mathbf{X}_{RD}$$
$$= \begin{bmatrix} -2{\cdot}1060413\text{E}-001 & 1{\cdot}9665720\text{E}-001 & 6{\cdot}2142025\text{E}-002 & 1{\cdot}6907682\text{E}-002 \\ 1{\cdot}8482239\text{E}-001 & 1{\cdot}0314957\text{E}-001 & -2{\cdot}7215087\text{E}-001 & -1{\cdot}4883910\text{E}-002 \\ 1{\cdot}1067568\text{E}-001 & -2{\cdot}1618738\text{E}-001 & 5{\cdot}9168543\text{E}-002 & 2{\cdot}4662662\text{E}-002 \\ 2{\cdot}0400201\text{E}-003 & -3{\cdot}7974458\text{E}-002 & -1{\cdot}1128691\text{E}-002 & 1{\cdot}8013300\text{E}-003 \end{bmatrix} \tag{66}$$

$$\mathbf{Y}_{LD} = \mathbf{Y}_{RD}$$
$$= \begin{bmatrix} 1{\cdot}9016880\text{E}-001 & -1{\cdot}9484099\text{E}-001 & -1{\cdot}9670089\text{E}-001 & -1{\cdot}5045991\text{E}-001 \\ -1{\cdot}6688872\text{E}-001 & -1{\cdot}0219695\text{E}-001 & 8{\cdot}6145113\text{E}-001 & 1{\cdot}3245055\text{E}-001 \\ -9{\cdot}9936609\text{E}-002 & 2{\cdot}1419080\text{E}-001 & -1{\cdot}8728880\text{E}-001 & -2{\cdot}1947077\text{E}-001 \\ -1{\cdot}8420730\text{E}-003 & 3{\cdot}7623749\text{E}-002 & 3{\cdot}5226137\text{E}-002 & -1{\cdot}6029870\text{E}-002 \end{bmatrix}, \tag{67}$$

$$\mathbf{Y}_{LD} = \mathbf{Y}_{RD}$$
$$= \begin{bmatrix} 7{\cdot}5273125\text{E}-001 & 6{\cdot}7561509\text{E}-001 & -2{\cdot}6240935\text{E}-001 & -3{\cdot}4867994\text{E}-002 \\ 7{\cdot}3388121\text{E}-001 & -7{\cdot}8551279\text{E}-001 & 3{\cdot}4671375\text{E}-001 & 4{\cdot}4497201\text{E}-002 \\ -9{\cdot}4888564\text{E}-002 & 3{\cdot}9996266\text{E}-001 & 9{\cdot}4883030\text{E}-001 & -6{\cdot}7877375\text{E}-002 \\ -7{\cdot}1600087\text{E}-003 & 6{\cdot}4667711\text{E}-002 & 5{\cdot}0393474\text{E}-002 & 9{\cdot}9753732\text{E}-001 \end{bmatrix} \tag{68}$$

8.2. AN ARBITRARY STRUCTURE-PRESERVING TRANSFORMATION

An arbitrary structure-preserving transformation is generated directly to demonstrate that it can be done. The following matrices will be found to transform the system according to equations (34)–(36) and to satisfy the constraints of structure preservation in equations (37)–(40).

$$\mathbf{W}_{LI} = \mathbf{W}_{RI}$$
$$= \begin{bmatrix} 5{\cdot}8082202\text{E}-001 & 1{\cdot}0606094\text{E}+000 & -7{\cdot}9348355\text{E}-001 & -1{\cdot}6982684\text{E}-003 \\ -1{\cdot}3353938\text{E}-001 & 5{\cdot}4275858\text{E}-004 & 4{\cdot}5278829\text{E}-001 & -2{\cdot}3212389\text{E}-001 \\ -4{\cdot}2391150\text{E}-002 & -2{\cdot}6496824\text{E}-001 & -7{\cdot}9187894\text{E}-001 & 5{\cdot}2685301\text{E}-001 \\ 8{\cdot}6497662\text{E}-003 & -1{\cdot}2107435\text{E}-002 & -1{\cdot}2584453\text{E}-002 & -8{\cdot}4738303\text{E}-002 \end{bmatrix}, \tag{69}$$

$$\mathbf{X}_{LI} = \mathbf{X}_{RI}$$
$$= \begin{bmatrix} -9{\cdot}1013738\text{E}-002 & 1{\cdot}2696448\text{E}-002 & -5{\cdot}6378918\text{E}-002 & 2{\cdot}5574612\text{E}-001 \\ 1{\cdot}8658541\text{E}-001 & 1{\cdot}0156098\text{E}+000 & -1{\cdot}7915443\text{E}-001 & -2{\cdot}7315635\text{E}-001 \\ -3{\cdot}1802849\text{E}-001 & -5{\cdot}5898336\text{E}-001 & -3{\cdot}6778518\text{E}-004 & 7{\cdot}6861594\text{E}-001 \\ -3{\cdot}6984937\text{E}-002 & -6{\cdot}1840484\text{E}-002 & -9{\cdot}0848782\text{E}-002 & 1{\cdot}9258083\text{E}-002 \end{bmatrix} \tag{70}$$

$$\mathbf{Y}_{LI} = \mathbf{Y}_{RI}$$
$$= \begin{bmatrix} 9{\cdot}0393835\mathrm{E}-002 & -4{\cdot}9401226\mathrm{E}-002 & 3{\cdot}7803968\mathrm{E}-001 & -5{\cdot}4316129\mathrm{E}-001 \\ 1{\cdot}2632639\mathrm{E}-001 & -5{\cdot}8275486\mathrm{E}-001 & -8{\cdot}5166588\mathrm{E}-001 & 5{\cdot}2878138\mathrm{E}-001 \\ 1{\cdot}4315547\mathrm{E}-001 & 1{\cdot}8690032\mathrm{E}-001 & 8{\cdot}3910214\mathrm{E}-001 & -1{\cdot}4456001\mathrm{E}+000 \\ -9{\cdot}4813229\mathrm{E}-003 & 1{\cdot}4693724\mathrm{E}-001 & 1{\cdot}9231113\mathrm{E}+000 & -5{\cdot}6421113\mathrm{E}-001 \end{bmatrix} \tag{71}$$

$$\mathbf{Z}_{LI} = \mathbf{Z}_{RI}$$
$$= \begin{bmatrix} 5{\cdot}5143953\mathrm{E}-001 & 9{\cdot}9066112\mathrm{E}-001 & -4{\cdot}7183597\mathrm{E}-001 & -2{\cdot}6369320\mathrm{E}-001 \\ -4{\cdot}4545625\mathrm{E}-001 & -9{\cdot}9611583\mathrm{E}-001 & 1{\cdot}5624943\mathrm{E}-001 & 7{\cdot}4259125\mathrm{E}-001 \\ -1{\cdot}1490259\mathrm{E}-002 & 5{\cdot}1678516\mathrm{E}-003 & 1{\cdot}2390942\mathrm{E}-001 & -4{\cdot}6207878\mathrm{E}-001 \\ 2{\cdot}9555283\mathrm{E}-001 & 5{\cdot}2165121\mathrm{E}-001 & 8{\cdot}7942317\mathrm{E}-002 & -8{\cdot}2296852\mathrm{E}-001 \end{bmatrix}. \tag{72}$$

Creating arbitrary structure-preserving transformations is relatively straightforward if the diagonalizing transformation is known. Work continues on methods for finding structure-preserving transformations as steps towards the diagonalizing transformation.

It is worth noting that the structure-preserving properties do not ensure that the transformed system matrices will be positive definite. In the present case, none of the system matrices in the transformed system, $\{\mathbf{K}', \mathbf{D}', \mathbf{M}'\}$, are positive definite. Despite this, the characteristic roots and system response will be computed accurately.

Matrix **K** *after the transformation*:

$$\begin{bmatrix} 2{\cdot}1712934\mathrm{E}-001 & 5{\cdot}2785217\mathrm{E}-001 & -2{\cdot}7825400\mathrm{E}-001 & 1{\cdot}1827036\mathrm{E}-001 \\ 5{\cdot}2785217\mathrm{E}-001 & 6{\cdot}7101092\mathrm{E}-001 & -6{\cdot}6523709\mathrm{E}-001 & 8{\cdot}3853412\mathrm{E}-002 \\ -2{\cdot}7825400\mathrm{E}-001 & -6{\cdot}6523709\mathrm{E}-001 & -2{\cdot}1152355\mathrm{E}+000 & 1{\cdot}1903511\mathrm{E}+000 \\ 1{\cdot}1827036\mathrm{E}-001 & 8{\cdot}3853412\mathrm{E}-002 & 1{\cdot}1903511\mathrm{E}+000 & -1{\cdot}7539227\mathrm{E}+000 \end{bmatrix} \tag{73}$$

Matrix **D** *after the transformation*:

$$\begin{bmatrix} 4{\cdot}2982498\mathrm{E}-001 & 6{\cdot}2870874\mathrm{E}-001 & -6{\cdot}7251128\mathrm{E}-001 & -5{\cdot}3743574\mathrm{E}-002 \\ 6{\cdot}2870874\mathrm{E}-001 & 6{\cdot}4565424\mathrm{E}-001 & -1{\cdot}1594473\mathrm{E}+000 & 1{\cdot}7679072\mathrm{E}-001 \\ -6{\cdot}7251128\mathrm{E}-001 & -1{\cdot}1594473\mathrm{E}+000 & 5{\cdot}5212756\mathrm{E}-002 & 1{\cdot}2353467\mathrm{E}+000 \\ -5{\cdot}3743574\mathrm{E}-002 & 1{\cdot}7679072\mathrm{E}-001 & 1{\cdot}2353467\mathrm{E}+000 & -1{\cdot}1773734\mathrm{E}+000 \end{bmatrix}, \tag{74}$$

Matrix **M** *after the transformation*:

$$\begin{bmatrix} 1{\cdot}3250830\mathrm{E}-001 & 2{\cdot}2376771\mathrm{E}-001 & -3{\cdot}0609591\mathrm{E}-001 & 3{\cdot}3731385\mathrm{E}-001 \\ 2{\cdot}2376771\mathrm{E}-001 & -7{\cdot}0041013\mathrm{E}-002 & -4{\cdot}4549000\mathrm{E}-001 & 5{\cdot}7181178\mathrm{E}-001 \\ -3{\cdot}0609591\mathrm{E}-001 & -4{\cdot}4549000\mathrm{E}-001 & 1{\cdot}6152710\mathrm{E}-001 & 8{\cdot}8852543\mathrm{E}-002 \\ 3{\cdot}3731385\mathrm{E}-001 & 5{\cdot}7181178\mathrm{E}-001 & 8{\cdot}8852543\mathrm{E}-002 & -9{\cdot}7504850\mathrm{E}-001 \end{bmatrix}. \tag{75}$$

8.3. A TRANSFORMATION TO "TRIDIAGONAL FORM" WHICH DOES NOT PRESERVE STRUCTURE

The origins of the transformations discussed in this paper lie in the use of Clifford Algebra as a tool for expressing the dynamics of second order systems [24]. Preservation of structure was not considered in reference [24] and if that is not a pre-requisite, it is attractive (and easy) to effect an initial transformation $\{\underline{\mathbf{T}_{L1}}, \underline{\mathbf{T}_{R1}}\}$, such that

$$\underline{\mathbf{T}}_{L1}^{\mathrm{T}}\begin{bmatrix}\mathbf{K} & \mathbf{0}\\ \mathbf{0} & -\mathbf{M}\end{bmatrix}\underline{\mathbf{T}_{R1}} = \begin{bmatrix}\mathbf{I}_{(N\times N)} & \mathbf{0}\\ \mathbf{0} & -\mathbf{I}_{(N\times N)}\end{bmatrix}, \qquad \underline{\mathbf{T}}_{L1}^{\mathrm{T}}\begin{bmatrix}\mathbf{0} & \mathbf{K}\\ \mathbf{K} & \mathbf{D}\end{bmatrix}\underline{\mathbf{T}_{R1}} = \begin{bmatrix}\mathbf{A}_1 & \mathbf{B}_1\\ \mathbf{C}_1 & \mathbf{R}_1\end{bmatrix}. \tag{76}$$

Evidently, this is not a structure-preserving transformation but it does preserve symmetry. Subsequent transformations, $\{\underline{\mathbf{T}_{Lk}}, \underline{\mathbf{T}_{Rk}}\}$, can then be carried out such that the cumulative transformation at every stage is

$$\underline{\mathbf{T}}_{Lk}^{\mathrm{T}}\begin{bmatrix}\mathbf{I}_{(N\times N)} & \mathbf{0}\\ \mathbf{0} & -\mathbf{I}_{(N\times N)}\end{bmatrix}\underline{\mathbf{T}_{Rk}} = \begin{bmatrix}\mathbf{I}_{(N\times N)} & \mathbf{0}\\ \mathbf{0} & -\mathbf{I}_{(N\times N)}\end{bmatrix},$$
$$\underline{\mathbf{T}}_{Lk}^{\mathrm{T}}\begin{bmatrix}\mathbf{A}_{(k-1)} & \mathbf{B}_{(k-1)}\\ \mathbf{C}_{(k-1)} & \mathbf{R}_{(k-1)}\end{bmatrix}\underline{\mathbf{T}_{Rk}} = \begin{bmatrix}\mathbf{A}_k & \mathbf{B}_k\\ \mathbf{C}_k & \mathbf{R}_k\end{bmatrix}. \tag{77}$$

Using extensions of some now standard methods in matrix analysis, matrices $\{\mathbf{A}_k, \mathbf{B}_k, \mathbf{C}_k, \mathbf{R}_k\}$ can be driven progressively towards tridiagonal form without any iteration — that is to say, the number of numerical operations utilized is fixed only by the dimensions of the system. The cumulative transformation which achieves this tridiagonalization is presented now for the example system. The transformation matrices are termed $\{\mathbf{W}_{TriDi}, \mathbf{X}_{TriDi}, \mathbf{Y}_{TriDi}, \mathbf{Z}_{TriDi}\}$. Because the system is symmetric, and all of the transformations preserve symmetry there is no need to distinguish between the left and right transformation matrices:

$$1{\cdot}0000000E+000$$
$$= \begin{bmatrix} 1{\cdot}1952286\mathrm{E}+000 & 0{\cdot}0000000\mathrm{E}+000 & 0{\cdot}0000000\mathrm{E}+000 & 0{\cdot}0000000\mathrm{E}+000\\ 0{\cdot}0000000\mathrm{E}+000 & 1{\cdot}0000000\mathrm{E}+000 & 0{\cdot}0000000\mathrm{E}+000 & 0{\cdot}0000000\mathrm{E}+000\\ 0{\cdot}0000000\mathrm{E}+000 & 0{\cdot}0000000\mathrm{E}+000 & -4{\cdot}0248662\mathrm{E}+001 & -3{\cdot}0186496\mathrm{E}-001\\ 0{\cdot}0000000\mathrm{E}+000 & 0{\cdot}0000000\mathrm{E}+000 & -2{\cdot}0124331\mathrm{E}-001 & -2{\cdot}6603710\mathrm{E}-001 \end{bmatrix}, \tag{78}$$

$$\mathbf{X}_{TriDi}$$
$$= \begin{bmatrix} 0.0000000\mathrm{E}+000 & 0.0000000\mathrm{E}+000 & 0.0000000\mathrm{E}+000 & 0.0000000\mathrm{E}+000\\ 0.0000000\mathrm{E}+000 & 0.0000000\mathrm{E}+000 & 0.0000000\mathrm{E}+000 & 0.0000000\mathrm{E}+000\\ 0.0000000\mathrm{E}+000 & 0.0000000\mathrm{E}+000 & -1.1436578\mathrm{E}-002 & -5.4654704\mathrm{E}-002\\ 0.0000000\mathrm{E}+000 & 0.0000000\mathrm{E}+000 & -5.7182889\mathrm{E}-003 & -1.1436578\mathrm{E}-002 \end{bmatrix}, \tag{79}$$

$$\mathbf{Y}_{TriDi} = \begin{bmatrix} 0{\cdot}0000000\text{E}+000 & 0{\cdot}0000000\text{E}+000 & 0{\cdot}0000000\text{E}+000 & 0{\cdot}0000000\text{E}+000 \\ 0{\cdot}0000000\text{E}+000 & 0{\cdot}0000000\text{E}+000 & -1{\cdot}8950854\text{E}-002 & 1{\cdot}1171348\text{E}-016 \\ 0{\cdot}0000000\text{E}+000 & 0{\cdot}0000000\text{E}+000 & 2{\cdot}9347540\text{E}-002 & 1{\cdot}7154867\text{E}-002 \\ 0{\cdot}0000000\text{E}+000 & 0{\cdot}0000000\text{E}+000 & -1{\cdot}0607221\text{E}-001 & -3{\cdot}4309733\text{E}-002 \end{bmatrix}, \tag{80}$$

$$\mathbf{Z}_{TriDi} = \begin{bmatrix} 1{\cdot}0000000\text{E}+000 & 0{\cdot}0000000\text{E}+000 & 0{\cdot}0000000\text{E}+000 & 0{\cdot}0000000\text{E}+000 \\ 0{\cdot}0000000\text{E}+000 & -7{\cdot}4535599\text{E}-001 & -6{\cdot}6693596\text{E}-001 & 4{\cdot}9651038\text{E}-016 \\ 0{\cdot}0000000\text{E}+000 & 5{\cdot}9628479\text{E}-001 & -6{\cdot}6556357\text{E}-001 & 4{\cdot}5013902\text{E}-001 \\ 0{\cdot}0000000\text{E}+000 & 2{\cdot}9814240\text{E}-001 & -3{\cdot}3621276\text{E}-001 & -9{\cdot}0027805\text{E}-001 \end{bmatrix}. \tag{81}$$

The final matrices $\{\mathbf{A}_3, \mathbf{B}_3, \mathbf{R}_3\}$ from equation (77) are

$$\mathbf{A}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -0.01895085380144 & 0 \\ 0 & -0{\cdot}01895085380144 & 0{\cdot}29647786986852 & -0{\cdot}25263348437028 \\ 0 & 0 & -0{\cdot}25263348437028 & -0{\cdot}20496038548204 \end{bmatrix}, \tag{82}$$

$$\mathbf{B}_3 = \begin{bmatrix} 0{\cdot}83666002653408 & 0 & 0 & 0 \\ 0 & -0{\cdot}74535599249993 & -0.66693596342097 & 0 \\ 0 & -1{\cdot}48919685962010 & 1{\cdot}69885063401876 & 0{\cdot}96946240184080 \\ 0 & 0 & 0{\cdot}00311023330480 & -2{\cdot}67923251869012 \end{bmatrix} \tag{83}$$

$$\mathrm{R}_3 = \begin{bmatrix} 0{\cdot}90000000000000 & 0{\cdot}67082039324994 & 0 & 0 \\ 0{\cdot}67082039324994 & 1{\cdot}72444444444444 & 0.27079877780442 & 0 \\ 0 & -0{\cdot}27079877780442 & 0{\cdot}25173559441080 & 0{\cdot}29752144238109 \\ 0 & 0 & 0{\cdot}29752144238109 & -0{\cdot}51533744553123 \end{bmatrix} \tag{84}$$

and $\mathbf{C}_3 = \mathbf{B}_3^{\mathrm{T}}$.

It is straightforward to demonstrate numerically that the characteristic roots of the transformed system are identical to the characteristic roots of the original one.

9. CONCLUSIONS

This paper discusses general co-ordinate transformations for second order systems. It notes that the full set of possible co-ordinate transformations for second order systems includes as a major subset the set of *structure-preserving* transformations. This set of transformations is defined concisely by equations (60)–(62) and it includes as a subset the set of all *first order* co-ordinate transformations which coincide with the view of co-ordinate transformations established in the structural dynamics community.

Within the set of *structure-preserving* co-ordinate transformations for (almost) any given system, there is a transformation involving only real numbers that transforms the original system into a new form in which the system matrices are real and diagonal. The only exceptions are defective systems. A route to the determination of this particular diagonalizing transformation is given in Appendix A beginning with solution of the well-known eigenvalue problem in complex numbers. Using this diagonalizing transformation, response in the time or frequency domain can be computed as the superposition of responses of N single-degree-of-freedom second order systems.

The implications of the paper are many. The use of structure-preserving transformations for second order systems may ultimately lead to improved computational performance in the determination of system characteristic behaviour—possibly through constructing the diagonalizing transformation as the product of a large number of elementary structure-preserving transformations. More importantly, though, it shows some prospects for substantially improved clarity in the study of generally damped systems.

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APPENDIX A: DERIVATION OF EQUATIONS (24) FROM A STATE-SPACE APPROACH

Begin with this form of the equation for the characteristic roots which implicitly requires that both **K** and **M** are non-singular. The case where either one is singular (or both are) can be dealt with by taking the limit:

$$\begin{bmatrix} \mathbf{E}_L & \mathbf{F}_L \\ \mathbf{G}_L & \mathbf{H}_L \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{E}_R & \mathbf{F}_R \\ \mathbf{G}_R & \mathbf{H}_R \end{bmatrix} = \begin{bmatrix} \mathbf{S}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_2 \end{bmatrix}, \tag{A.1}$$

$$\begin{bmatrix} \mathbf{E}_L & \mathbf{F}_L \\ \mathbf{G}_L & \mathbf{H}_L \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{E}_R & \mathbf{F}_R \\ \mathbf{G}_R & \mathbf{H}_R \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{(N\times N)} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{(N\times N)} \end{bmatrix}. \tag{A.2}$$

Equations (A.1) and (A.2) find the *left* characteristic vectors as well as the *right*. Matrices $\mathbf{S}_1$ and $\mathbf{S}_2$ are diagonal. Where complex roots occur, they occur in conjugate pairs. If $2P$ of the $2N$ roots are complex, arrange the roots and vectors such that $\mathbf{S}_2(k,k) = conj(\mathbf{S}_1(k,k))$ for $k \leqslant P$. The remaining $2Q$ (with $Q = N - P$) roots are real.

A.1 . SELF-ADJOINT SYSTEMS WITH **M** POSITIVE DEFINITE AND BOTH **D** AND **K** POSITIVE SEMI-DEFINITE

Consider, initially, the class of self-adjoint systems with positive definite **M** and positive semi-definite **D** and **K**. For these systems, the real roots are all negative (or zero) [1,23] and they occur in two distinct groups. For one half of the real roots, the associated right vectors (columns $(P+1{:}N)$ of $\mathbf{E}_R$ and $\mathbf{G}_R$) comprise purely real entries when the scaling of equation (A.2) is applied. For the other half of the real roots, the associated right vectors (columns $(P+1{:}N)$ of $\mathbf{F}_R$ and $\mathbf{H}_R$) comprise purely imaginary entries. Similar statements apply to the left vectors. Define the following useful matrix:

$$\mathbf{J}=\begin{bmatrix} \frac{1}{\sqrt{2}}\mathbf{I}_{(P\times P)} & \mathbf{0} & \frac{-j}{\sqrt{2}}\mathbf{I}_{(P\times P)} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{(Q\times Q)} & \mathbf{0} & \mathbf{0} \\ \frac{1}{\sqrt{2}}\mathbf{I}_{(P\times P)} & \mathbf{0} & \frac{j}{\sqrt{2}}\mathbf{I}_{(P\times P)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & j\mathbf{I}_{(Q\times Q)} \end{bmatrix}, \qquad \mathbf{J}^{\mathrm{T}}\mathbf{J}=\begin{bmatrix} \mathbf{I}_{(N\times N)} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_{(N\times N)} \end{bmatrix}. \tag{A.3}$$

Post-multiply equations (A.1) and (A.2) by **J** and pre-multiply by $\mathbf{J}^{\mathrm{T}}$. All imaginary components are eliminated from the equations by this action and the real matrices $\{\mathbf{W}'_{LD}, \mathbf{X}'_{LD}, \mathbf{Y}'_{LD}, \mathbf{Z}'_{LD}\}$, $\{\mathbf{W}'_{RD}, \mathbf{W}'_{RD}, \mathbf{Y}'_{RD}, \mathbf{Z}'_{RD}\}$, and $\{\mathbf{S}_x, \mathbf{S}_y, \mathbf{S}_z\}$ are obtained with $\{\mathbf{S}_x, \mathbf{S}_y, \mathbf{S}_z\}$ diagonal:

$$\begin{bmatrix} \mathbf{W}'_{LD} & \mathbf{X}'_{LD} \\ \mathbf{Y}'_{LD} & \mathbf{Z}'_{LD} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{W}'_{RD} & \mathbf{X}'_{RD} \\ \mathbf{Y}'_{RD} & \mathbf{Z}'_{RD} \end{bmatrix} = \begin{bmatrix} \mathbf{S}_x & \mathbf{S}_y \\ \mathbf{S}_y & \mathbf{S}_z \end{bmatrix}, \tag{A.4}$$

$$\begin{bmatrix} \mathbf{W}'_{LD} & \mathbf{X}'_{LD} \\ \mathbf{Y}'_{LD} & \mathbf{Z}'_{LD} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{W}'_{RD} & \mathbf{X}'_{RD} \\ \mathbf{Y}'_{RD} & \mathbf{Z}'_{RD} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{(N\times N)} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_{(N\times N)} \end{bmatrix} \tag{A.5}$$

The primes are used here because left and right versions of matrices **W**, **X**, **Y**, **Z** will be defined differently in due course. Although there is actually no difference between the left and right versions of these matrices in the case of the class of second order systems being considered at present, the distinction between the two is maintained for use in the more general cases.

Note the following about the contents of $\mathbf{S}_x$, $\mathbf{S}_y$ and $\mathbf{S}_z$:

$$\begin{aligned} &\mathbf{S}_x(k,k)=\mathrm{real}(\mathbf{S}_1(k,k)) \quad \forall k\leqslant P, && \mathbf{S}_x(k,k)=\mathbf{S}_1(k,k) \quad \forall k>P, \\ &\mathbf{S}_y(k,k)=\mathrm{imag}(\mathbf{S}_1(k,k)) \quad \forall k\leqslant P, && \mathbf{S}_y(k,k)=0{\cdot}0 \quad \forall k>P, \\ &\mathbf{S}_z(k,k)=\mathrm{real}(\mathbf{S}_1(k,k)) \quad \forall k\leqslant P, && \mathbf{S}_z(k,k)=-\mathbf{S}_2(k,k) \quad \forall k>P. \end{aligned} \tag{A.6}$$

Compute a real diagonal $(N\times N)$ matrix, γ, such that

$$\begin{bmatrix} \cosh(\gamma) & \sinh(\gamma) \\ \sinh(\gamma) & \cosh(\gamma) \end{bmatrix} \begin{bmatrix} \mathbf{S}_x & \mathbf{S}_y \\ \mathbf{S}_y & \mathbf{S}_z \end{bmatrix} \begin{bmatrix} \cosh(\gamma) & \sinh(\gamma) \\ \sinh(\gamma) & \cosh(\gamma) \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \Omega \\ \Omega & 2\zeta\Omega \end{bmatrix} \tag{A.7}$$

in which $\mathbf{\Omega}$ and $2\zeta\mathbf{\Omega}$ are also real diagonal $(N\times N)$ matrices. This notation is chosen deliberately to invoke a connection with the quantities ω_n and $2\zeta\omega_n$ commonly used to define a unit-mass single-degree-of-freedom second order system.

Each diagonal entry of γ can be computed separately. If the kth pair of roots is a complex conjugate pair $(k\leqslant P)$, then $\mathbf{S}_z(k,k)+\mathbf{S}_x(k,k)=0$ and

$$\gamma(k,k)=\tfrac{1}{2}\sinh^{-1}\left(\frac{\mathbf{S}_z(k,k)-\mathbf{S}_x(k,k)}{2\mathbf{S}_y(k,k)}\right). \tag{A.8}$$

If the kth pair of roots is a pair of real roots ($k > P$), then $\mathbf{S}_y(k, k) = 0$ and the expression for $\gamma(k,k)$ is

$$\gamma(k, k) = \tfrac{1}{2}\cosh^{-1}\left(\frac{\mathbf{S}_z(k, k) - \mathbf{S}_x(k, k)}{\mathbf{S}_z(k, k) + \mathbf{S}_x(k, k)}\right) = \tfrac{1}{2}\cosh^{-1}\left(\frac{\mathbf{S}_2(k, k) + \mathbf{S}_1(k, k)}{\mathbf{S}_2(k, k) - \mathbf{S}_1(k, k)}\right). \qquad \text{(A.9)}$$

In this case, for $\gamma(k,k)$ to be a real number, it is necessary that the operand of the $\cosh^{-1}(.)$ function is greater than unity. For the class of symmetric second order systems in which $\mathbf{M}$ is positive definite and $\mathbf{D}$ and $\mathbf{K}$ are positive semi-definite, all real roots are negative and therefore the magnitude of the operand is necessarily greater than unity. This reasoning alone does not guarantee that the sign of the operand is positive. Experience shows that the ordering of the real roots described above always produces a positive operand. Since the general case of second order systems will be dealt with shortly, it suffices to leave this remark without further justification.

Now define (right and left versions of) $\mathbf{W}$, $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ based on $\mathbf{W}'$, $\mathbf{X}'$, $\mathbf{Y}'$ and $\mathbf{Z}'$ and the diagonal matrices $\boldsymbol{\gamma}$, $\boldsymbol{\Omega}$ and $2\boldsymbol{\zeta}\boldsymbol{\Omega}$

$$\begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} = \begin{bmatrix} \mathbf{W}'_R & \mathbf{X}'_R \\ \mathbf{Y}'_R & \mathbf{Z}'_R \end{bmatrix} \begin{bmatrix} \cosh(\gamma) & \sinh(\gamma) \\ \sinh(\gamma) & \cosh(\gamma) \end{bmatrix} \begin{bmatrix} \boldsymbol{\Omega} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \qquad \text{(A.10)}$$

$$\begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix} = \begin{bmatrix} \mathbf{W}'_L & \mathbf{X}'_L \\ \mathbf{Y}'_L & \mathbf{Z}'_L \end{bmatrix} \begin{bmatrix} \cosh(\gamma) & \sinh(\gamma) \\ \sinh(\gamma) & \cosh(\gamma) \end{bmatrix} \begin{bmatrix} \boldsymbol{\Omega} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}. \qquad \text{(A.11)}$$

Equations (15) and (16) in the main body of the paper follow naturally with

$$\mathbf{K}_D := \boldsymbol{\Omega}^2, \qquad \mathbf{D}_D := (2\boldsymbol{\zeta}\boldsymbol{\Omega}), \qquad \mathbf{M}_D := \mathbf{I}. \qquad \text{(A.12)}$$

A.2 . GENERAL SELF-ADJOINT SYSTEMS

In the case of general self-adjoint systems, it is attractive to ensure that the left and right transformations are identical $\left(\underline{\mathbf{U}_L} = \underline{\mathbf{U}_R}\right)$. In order to achieve this, a more general definition of $\mathbf{J}$ is needed:

$$\mathbf{J} = \begin{bmatrix} \frac{1}{\sqrt{2}}\mathbf{I}_{(P\times P)} & \mathbf{0} & \frac{-j}{\sqrt{2}}\mathbf{I}_{(P\times P)} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_1 & \mathbf{0} & \mathbf{0} \\ \frac{1}{\sqrt{2}}\mathbf{I}_{(P\times P)} & \mathbf{0} & \frac{j}{\sqrt{2}}\mathbf{I}_{(P\times P)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{L}_2 \end{bmatrix}, \qquad \text{(A.13)}$$

where $\{\mathbf{L}_1, \mathbf{L}_2\}$ are diagonal matrices defined shortly. As before, the $2P$ complex roots and associated vectors are arranged such that $\mathbf{S}_2(k,k) = conj(\mathbf{S}_1(k,k))$ for $k \leqslant P$. The remaining $2Q$ (with $Q = N - P$) roots are real.

The real roots must be collected into pairs. There is some degree of freedom in this pairing although, as above, there are some constraints. For self-adjoint systems in general, it is always possible to establish a pairing such that the real roots fall into three different categories (A, B, C) of the six possible categories defined in Table A1.

When the system is self-adjoint, there is no difference between left and right eigenvectors but even in the general case, it is readily seen that if the right eigenvector associated with a given real root is purely imaginary, the same must be true of the left eigenvector. Similarly for purely real left and right eigenvectors.

Let $\{Q_A, Q_B, Q_C\}$ represent the number of pairs of real roots in each of the categories A, B, C respectively.

TABLE A1

Categories of pairs of real roots

Category	Product of the 2 roots	Eigenvectors for root 1	Eigenvectors for root 2
A	Positive	Real	Imaginary
B	Negative	Real	Real
C	Negative	Imaginary	Imaginary
D	Positive	Real	Real
E	Positive	Imaginary	Imaginary
F	Negative	Real	Imaginary

Then define $\mathbf{L}_1$ and $\mathbf{L}_2$ as follows :

$$\mathbf{L}_1 := \begin{bmatrix} \mathbf{I}_{QA} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{QB} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & j\mathbf{I}_{QC} \end{bmatrix}, \qquad \mathbf{L}_2 := \begin{bmatrix} j\mathbf{I}_{QA} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{QB} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & j\mathbf{I}_{QC} \end{bmatrix}. \tag{A.14}$$

Post-multiply equations (A.1) and (A.2) by $\mathbf{J}$ (from equation (A.13)) and pre-multiply by $\mathbf{J}^{\mathrm{T}}$. All imaginary components are eliminated from the equations by this action and the real matrices $\{\mathbf{W}'_{LD}, \mathbf{X}'_{LD}, \mathbf{Y}'_{LD}, \mathbf{Z}'_{LD}\}$, $\{\mathbf{W}'_{RD}, \mathbf{W}'_{RD}, \mathbf{Y}'_{RD}, \mathbf{Z}'_{RD}\}$, and $\{\mathbf{S}_x, \mathbf{S}_y, \mathbf{S}_z\}$ are obtained with $\{\mathbf{S}_x, \mathbf{S}_y, \mathbf{S}_z\}$ diagonal. These matrices obey

$$\begin{bmatrix} \mathbf{W}'_{LD} & \mathbf{X}'_{LD} \\ \mathbf{Y}'_{LD} & \mathbf{Z}'_{LD} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{W}'_{RD} & \mathbf{X}'_{RD} \\ \mathbf{Y}'_{RD} & \mathbf{Z}'_{RD} \end{bmatrix} = \begin{bmatrix} \mathbf{S}_x & \mathbf{S}_y \\ \mathbf{S}_y & \mathbf{S}_z \end{bmatrix}, \tag{A.15}$$

$$\begin{bmatrix} \mathbf{W}'_{LD} & \mathbf{X}'_{LD} \\ \mathbf{Y}'_{LD} & \mathbf{Z}'_{LD} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{W}'_{RD} & \mathbf{X}'_{RD} \\ \mathbf{Y}'_{RD} & \mathbf{Z}'_{RD} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{(N\times N)} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_{(N\times N)} \end{bmatrix}, \tag{A.16}$$

where $\mathbf{G}_1$ and $\mathbf{G}_2$ are diagonal matrices whose diagonal entries are all either unity or its negative. The primes are used again here because left and right versions of matrices **W**, **X**, **Y**, **Z** will be defined differently in due course. Also, the distinction between left and right versions is maintained for use in the more general cases.

One further transformation is required from this point. This involves finding real diagonal matrices $\{\mathbf{N}_{11}, \mathbf{N}_{12}, \mathbf{N}_{21}, \mathbf{N}_{22}\}$ satisfying

$$\begin{bmatrix} \mathbf{N}_{11} & \mathbf{N}_{12} \\ \mathbf{N}_{21} & \mathbf{N}_{22} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{S}_x & \mathbf{S}_y \\ \mathbf{S}_y & \mathbf{S}_z \end{bmatrix} \begin{bmatrix} \mathbf{N}_{11} & \mathbf{N}_{12} \\ \mathbf{N}_{21} & \mathbf{N}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{K}_D \\ \mathbf{K}_D & \mathbf{D}_D \end{bmatrix}, \tag{A.17}$$

$$\begin{bmatrix} \mathbf{N}_{11} & \mathbf{N}_{12} \\ \mathbf{N}_{21} & \mathbf{N}_{22} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{G}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_2 \end{bmatrix} \begin{bmatrix} \mathbf{N}_{11} & \mathbf{N}_{12} \\ \mathbf{N}_{21} & \mathbf{N}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{K}_D & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_D \end{bmatrix}, \tag{A.18}$$

for some real diagonal matrices $\{\mathbf{K}_D, \mathbf{D}_D, \mathbf{M}_D\}$. This problem decouples into N distinct quadratic problems involving (2×2) matrices.

The first P decoupled problems can be addressed by posing them initially in the form of equation (A.7), finding solutions to that in the form of equation (A.8) and then scaling as equation (A.10) indicates.

The next Q_A decoupled problems can be addressed by posing them initially in the form of equation (A.7), finding solutions to that in the form of equation (A.9) and then scaling as equation (A.10) indicates.

The next Q_B decoupled problems involve determining $\{\mathbf{N}_{11}(k,k),\ \mathbf{N}_{12}(k,k),\ \mathbf{N}_{21}(k,k),\ \mathbf{N}_{22}(k,k)\}$ such that

$$\begin{bmatrix} \mathbf{N}_{11}(k,k) & \mathbf{N}_{12}(k,k) \\ \mathbf{N}_{21}(k,k) & \mathbf{N}_{22}(k,k) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{S}_x(k,k) & 0 \\ 0 & \mathbf{S}_z(k,k) \end{bmatrix} \begin{bmatrix} \mathbf{N}_{11}(k,k) & \mathbf{N}_{12}(k,k) \\ \mathbf{N}_{21}(k,k) & \mathbf{N}_{22}(k,k) \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{K}_D(k,k) \\ \mathbf{K}_D(k,k) & \mathbf{D}_D(k,k) \end{bmatrix}, \tag{A.19}$$

$$\begin{bmatrix} \mathbf{N}_{11}(k,k) & \mathbf{N}_{12}(k,k) \\ \mathbf{N}_{21}(k,k) & \mathbf{N}_{22}(k,k) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{N}_{11}(k,k) & \mathbf{N}_{12}(k,k) \\ \mathbf{N}_{21}(k,k) & \mathbf{N}_{22}(k,k) \end{bmatrix} = \begin{bmatrix} \mathbf{K}_D(k,k) & 0 \\ 0 & -\mathbf{M}_D(k,k). \end{bmatrix} \tag{A.20}$$

Solutions to this problem are possible in the form

$$\begin{bmatrix} \mathbf{N}_{11}(k,k) & \mathbf{N}_{12}(k,k) \\ \mathbf{N}_{21}(k,k) & \mathbf{N}_{22}(k,k) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}, \tag{A.21}$$

where θ is determined from

$$(\mathbf{S}_x(k,k) + \mathbf{S}_z(k,k)) + (\mathbf{S}_x(k,k) - \mathbf{S}_z(k,k))\cos(2\theta) = 0 \tag{A.22}$$

and selection of a is obvious. For roots in this category, the angle θ determined from equation (A.22) is always real because $\mathbf{S}_x(k,k)$ and $\mathbf{S}_z(k,k)$ always have opposite sign. After determination of a and θ, $\mathbf{K}_D(k,k)$ and $\mathbf{D}_D(k,k)$ are calculated easily and $\mathbf{M}_D(k,k) = -1$.

The final Q_C decoupled problems involve determining $\{\mathbf{N}_{11}(k,k),\ \mathbf{N}_{12}(k,k),\ \mathbf{N}_{21}(k,k),\ \mathbf{N}_{22}(k,k)\}$ such that

$$\begin{bmatrix} \mathbf{N}_{11}(k,k) & \mathbf{N}_{12}(k,k) \\ \mathbf{N}_{21}(k,k) & \mathbf{N}_{22}(k,k) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{S}_x(k,k) & 0 \\ 0 & \mathbf{S}_z(k,k) \end{bmatrix} \begin{bmatrix} \mathbf{N}_{11}(k,k) & \mathbf{N}_{12}(k,k) \\ \mathbf{N}_{21}(k,k) & \mathbf{N}_{22}(k,k) \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{K}_D(k,k) \\ \mathbf{K}_D(k,k) & \mathbf{D}_D(k,k) \end{bmatrix}, \tag{A.23}$$

$$\begin{bmatrix} \mathbf{N}_{11}(k,k) & \mathbf{N}_{12}(k,k) \\ \mathbf{N}_{21}(k,k) & \mathbf{N}_{22}(k,k) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \mathbf{N}_{11}(k,k) & \mathbf{N}_{12}(k,k) \\ \mathbf{N}_{21}(k,k) & \mathbf{N}_{22}(k,k) \end{bmatrix} = \begin{bmatrix} \mathbf{K}_D(k,k) & 0 \\ 0 & -\mathbf{M}_D(k,k) \end{bmatrix}. \tag{A.24}$$

Solutions to this problem are possible in the form of equation (A.21) using θ as defined equation (A.22) and selection of a is obvious. Again θ determined from equation (A.22) is always real because $\mathbf{S}_x(k,k)$ and $\mathbf{S}_z(k,k)$ always have opposite sign. $\mathbf{K}_D(k,k)$ and $\mathbf{D}_D(k,k)$ are calculated easily and in this case $\mathbf{M}_D(k,k) = 1$.

A.3 . GENERAL SECOND-ORDER SYSTEMS

In the case of general second order systems, it is not necessarily possible to proceed from the solution of equations (A.1) and (A.2) using further transformations which are symmetric. The eigenvectors associated with the complex roots are inherently amenable to symmetric treatment but those associated with real roots are not.

For this reason, it is necessary in the general case to define left and right versions of the matrix $\mathbf{J}$ as

$$\mathbf{J}_L = \begin{bmatrix} \frac{1}{\sqrt{2}}\mathbf{I}_{(P\times P)} & \mathbf{0} & \frac{-j}{\sqrt{2}}\mathbf{I}_{(P\times P)} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_{L1} & \mathbf{0} & \mathbf{0} \\ \frac{1}{\sqrt{2}}\mathbf{I}_{(P\times P)} & \mathbf{0} & \frac{j}{\sqrt{2}}\mathbf{I}_{(P\times P)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{L}_{L2} \end{bmatrix}, \quad \mathbf{J}_R = \begin{bmatrix} \frac{1}{\sqrt{2}}\mathbf{I}_{(P\times P)} & \mathbf{0} & \frac{-j}{\sqrt{2}}\mathbf{I}_{(P\times P)} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_{R1} & \mathbf{0} & \mathbf{0} \\ \frac{1}{\sqrt{2}}\mathbf{I}_{(P\times P)} & \mathbf{0} & \frac{j}{\sqrt{2}}\mathbf{I}_{(P\times P)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{L}_{R2} \end{bmatrix}. \tag{A.25}$$

As before, the $2P$ complex roots and associated vectors are arranged such that $\mathbf{S}_2(k,k) = conj(\boldsymbol{S}_1(k,k))$ for $k \leqslant P$. The remaining $2Q$ (with $Q = N-P$) roots are real. The real roots must be collected into pairs. Each of these pairs must fall into one of the six different categories described in Table A1.

Let $\{Q_A, Q_B, Q_C, Q_D, Q_E, Q_F\}$ represent the number of pairs of real roots in each of the categories *A,B,C,D,E,F* respectively and define $\{\mathbf{L}_{L1}, \mathbf{L}_{L2}, \mathbf{L}_{R1}, \mathbf{L}_{R2}\}$ as follows:

$$\text{diag}(\mathbf{L}_{L1}) := \begin{bmatrix} \mathbf{E}_{QA} \\ \mathbf{E}_{QB} \\ j\mathbf{E}_{QC} \\ \mathbf{E}_{QD} \\ -j\mathbf{E}_{QE} \\ \mathbf{E}_{QF} \end{bmatrix}, \quad \text{diag}(\mathbf{L}_{L2}) := \begin{bmatrix} j\mathbf{E}_{QA} \\ \mathbf{E}_{QB} \\ j\mathbf{E}_{QC} \\ -\mathbf{E}_{QD} \\ j\mathbf{E}_{QE} \\ -j\mathbf{E}_{QF} \end{bmatrix},$$

$$\text{diag}(\mathbf{L}_{R1}) := \begin{bmatrix} \mathbf{E}_{QA} \\ \mathbf{E}_{QB} \\ j\mathbf{E}_{QC} \\ \mathbf{E}_{QD} \\ j\mathbf{E}_{QE} \\ \mathbf{E}_{QF} \end{bmatrix}, \quad \text{diag}(\mathbf{L}_{R2}) := \begin{bmatrix} j\mathbf{E}_{QA} \\ \mathbf{E}_{QB} \\ j\mathbf{E}_{QC} \\ \mathbf{E}_{QD} \\ j\mathbf{E}_{QE} \\ j\mathbf{E}_{QF} \end{bmatrix}, \tag{A.26}$$

where $\{\mathbf{E}_{QA}, \mathbf{E}_{QB}, \mathbf{E}_{QC}, \mathbf{E}_{QD}, \mathbf{E}_{QE}, \mathbf{E}_{QF}\}$ are column vectors with unit values in each entry and having dimensions $\{Q_A, Q_B, Q_C, Q_D, Q_E, Q_F\}$ respectively.

Post-multiply equations (A.1) and (A.2) by $\mathbf{J}_R$ (from equation (A.25)) and pre-multiply by $\mathbf{J}_L^T$. All imaginary components are eliminated from the equations by this action and the real matrices $\{\mathbf{W}'_{LD}, \mathbf{X}'_{LD}, \mathbf{Y}'_{LD}, \mathbf{Z}'_{LD}\}$, $\{\mathbf{W}'_{RD}, \mathbf{W}'_{RD}, \mathbf{Y}'_{RD}, \mathbf{Z}'_{RD}\}$, and $\{\mathbf{S}_x, \mathbf{S}_y, \mathbf{S}_z\}$ are obtained with $\{\mathbf{S}_x, \mathbf{S}_y, \mathbf{S}_z\}$ diagonal. These matrices obey equations (A.15) and (A.16) with $\mathbf{G}_1$ and $\mathbf{G}_2$ being diagonal matrices. As before, each diagonal entry of $\mathbf{G}_1$ is either unity or its negative. The same applies to $\mathbf{G}_2$.

One further transformation is required from this point. This involves finding real diagonal matrices $\{\mathbf{N}_{11}, \mathbf{N}_{12}, \mathbf{N}_{21}, \mathbf{N}_{22}\}$ satisfying equations (A.17) and (A.18) for some real diagonal matrices $\{\mathbf{K}_D, \mathbf{D}_D, \mathbf{M}_D\}$. This problem decouples into N distinct quadratic problems involving (2×2) matrices.

Methods for finding solutions to the first $(P + Q_A + Q_B + Q_C)$ of those decoupled problems have been presented above.

The Q_D decoupled problems arising for the category D pairs of real roots present themselves in the form

$$\begin{bmatrix} \mathbf{N}_{11}(k,k) & \mathbf{N}_{12}(k,k) \\ \mathbf{N}_{21}(k,k) & \mathbf{N}_{22}(k,k) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{S}_x(k,k) & 0 \\ 0 & \mathbf{S}_z(k,k) \end{bmatrix} \begin{bmatrix} \mathbf{N}_{11}(k,k) & \mathbf{N}_{12}(k,k) \\ \mathbf{N}_{21}(k,k) & \mathbf{N}_{22}(k,k) \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{K}_D(k,k) \\ \mathbf{K}_D(k,k) & \mathbf{D}_D(k,k) \end{bmatrix}, \tag{A.27}$$

$$\begin{bmatrix} \mathbf{N}_{11}(k,k) & \mathbf{N}_{12}(k,k) \\ \mathbf{N}_{21}(k,k) & \mathbf{N}_{22}(k,k) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \mathbf{N}_{11}(k,k) & \mathbf{N}_{12}(k,k) \\ \mathbf{N}_{21}(k,k) & \mathbf{N}_{22}(k,k) \end{bmatrix} = \begin{bmatrix} \mathbf{K}_D(k,k) & 0 \\ 0 & -\mathbf{M}_D(k,k) \end{bmatrix}. \tag{A.28}$$

with the product $(\mathbf{S}_x(k,k)\mathbf{S}_z(k,k))$ being negative. Real solutions to this problem are available in the form

$$\begin{bmatrix} \mathbf{N}_{11}(k,k) & \mathbf{N}_{12}(k,k) \\ \mathbf{N}_{21}(k,k) & \mathbf{N}_{22}(k,k) \end{bmatrix} = \begin{bmatrix} \cosh(\gamma) & \sinh(\gamma) \\ \sinh(\gamma) & \cosh(\gamma) \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}, \tag{A.29}$$

where the expression for γ is available from equation (A.9) and a is trivial to find thereafter. $\mathbf{K}_D(k,k)$ and $\mathbf{D}_D(k,k)$ are then calculated easily and $\mathbf{M}_D(k,k) = 1$.

The Q_E decoupled problems arising for the category E pairs of real roots also present themselves in the form of equations (A.27) and (A.28) and the same solution approach (equations (A.29) and (A.9)) is appropriate. Here, the location of the minus signs in equation (A.26) is chosen deliberately such that $\mathbf{M}_D(k,k) = 1$.

The final Q_{F} decoupled problems arising for the category F pairs of real roots present themselves in the form of equations (A.19) and (A.20). Solutions are obtainable in the form of equations (A.21) and (A.22) may be used to determine the appropriate value of θ. Once again, the location of the minus signs in equation (A.26) is chosen deliberately such that $\mathbf{M}_D(k,k) = 1$ for the pairs of real roots in this category.

Three final remarks are appropriate in Appendix A.

Firstly, the case of singular $\mathbf{M}$ or $\mathbf{K}$ requires further treatment. In all of the above, the scaling of columns of $\{\underline{\mathbf{U}_L}, \underline{\mathbf{U}_R}\}$ has been such that the diagonal entries of $\mathbf{M}_D$ are all either unity or its negative. In fact, the scaling of these columns needs to be released for complete generality. This is especially necessary in order that the case where $\mathbf{M}$ is singular can be dealt with. The following scaling rule is a good candidate for all situations except those in one exceptionally unlikely set.

$$\mathbf{M}_D^2 + \mathbf{D}_D^2 f^{-2} + \mathbf{K}_D^2 f^{-4} = \mathbf{I}, \tag{A.30}$$

where f is any arbitrary real scalar frequency (rad/s). This scaling has the considerable attraction that it approaches "mass-normalization" when $\mathbf{M}$ is positive definite as $f \to \infty$.

The only situations in which this is unsatisfactory are those where there are some real left and right vectors, $\{\mathbf{v}_R, \mathbf{v}_L\}$ such that

$$\begin{aligned} \mathbf{M}_D\mathbf{v}_R = \mathbf{D}_D\mathbf{v}_R = \mathbf{K}_D\mathbf{v}_R = \mathbf{0}, \\ \mathbf{v}_L^{\mathrm{T}}\mathbf{M}_D = \mathbf{v}_L^{\mathrm{T}}\mathbf{D}_D = \mathbf{v}_L^{\mathrm{T}}\mathbf{K}_D = \mathbf{0}. \end{aligned} \tag{A.31}$$

If a scaling derived using equation (A.30) suggests extremely large scaling factors for the vectors, then it would obviously be sensible to check that equation (A.31) does not actually apply. If equation (A.31) does apply for some $\{\mathbf{v}_R, \mathbf{v}_L\}$, then the model evidently includes some completely redundant degrees of freedom which have no dynamic stiffness at any frequency and it is not difficult to deflate the model to remove these. The set of second order systems which satisfy equation (A.31) is so small that this possibility is dismissed apart from that check.

Now consider the case where **M** is singular. Solutions to equations (A.1) and (A.2) cannot then be found. However, replacing **M** by $(\mathbf{M}+\varepsilon\Delta\mathbf{M})$ enables to derive solutions provided that $(\mathbf{M}+\varepsilon\Delta\mathbf{M})$ is non-singular for all ε in some range. Finding the diagonalizing transformation for numerous different values of ε using equation (A.3) onwards and subsequently scaling the solutions such that equation (A.30) is satisfied yields the scaled diagonalizing transformation as a smooth function of ε. The value of this transformation for $\varepsilon = 0$ can then be deduced.

A similar procedure can apply for the case where **K** is singular. Where both **K** and **M** are singular, replace {**K**, **M**} by $\{(\mathbf{K}+\varepsilon\Delta\mathbf{K}), (\mathbf{M}+\varepsilon\Delta\mathbf{M})\}$ respectively.

Evidently, therefore, the diagonalizing transformation can be derived for any systems which are not defective and which cannot satisfy equation (A.31).

The second remark concerns symmetry and $\{\underline{\mathbf{U}_L}, \underline{\mathbf{U}_R}\}$. The general procedure outlined after equation (A.24) provides for the determination of a real diagonalizing transformation for any second order system which is not defective. The procedure for determining the diagonalizing transformation for self-adjoint systems asserts without proof that the real roots can be paired such that the pairs all lie in categories A, B or C (cf., Table A1) and it shows that for these categories of pairs of real roots, symmetric transformations can be used to retain the symmetry inherent in the original solution for the complex roots and eigenvectors. That statement is supported by experience of a large number of cases but a formal proof is considered beyond the scope of this paper. In a general purpose algorithm for determining a diagonalizing transformation for a second order system, it is sensible to pair the real roots such that as many as possible of the pairs of real roots lie in categories A, B or C in order that the transformation resulting will return $\underline{\mathbf{U}_L} = \underline{\mathbf{U}_R}$ whenever the system is self-adjoint.

Finally, for all of the pairs of complex roots and for the pairs of real roots occurring in classes A, D and E, (cf., Table A1), the transformation from the original eigenvalue-eigenvector form to the diagonalizing transformation form involves the use of a (2×2) matrix having $\cosh(\gamma)$ for both diagonal entries and $\sinh(\gamma)$ for both off-diagonal entries. The condition of this matrix becomes very poor as γ approaches large positive or negative values. This in turn happens whenever a pair of real roots is very close together or whenever a pair of complex roots are close together. If some parameter is varied such that a pair of complex roots draw together and eventually become a pair of real roots, the diagonalizing transformation will be found to vary smoothly throughout. Experience suggests therefore that the poor condition of the transformation between the conventional solution and the diagonalizing transformation is, in fact, a reflection of the particular inappropriateness of the conventional solution method for systems having pairs of roots close to the real-complex border.